Resonance phenomenon in a forced damped oscillator

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The ratio of the viscosity of water to air was measured in this experiment using a Driven Harmonic Motion Analyzer. The amplitude of an oscillating 50g mass was measured in air and in water for various driving frequencies. The resonant amplitudes for the mass in air and water were found to be 156 ± 2 mm and 76 ± 2 mm respectively. An estimate for the viscosity of air was 0.12 ± 0.006 Ns/m and that of water was 0.24 ± 0.002 Ns/m. The resulting ratio of the viscosities was found to be 2.0 ± 0.1. The published value for the same is 55.56.

INTRODUCTION:

A child using a swing realizes that if the pushes are properly timed, the swing can move with a very large amplitude. The driving force, in this case the agent pushing the swing, exactly replenishes the energy that the system loses if its frequency equals the natural frequency of the system. This phenomenon is called resonance. More formally, it is the maximum response of a system when its natural frequency becomes equal to the forcing frequency.

An example of resonance occurred in the Tacoma Narrows Bridge in Washington State in 1940. The wind blowing through the Tacoma Narrows broke up into vortices which provided puffs of wind that shook the bridge at a frequency that matched one of its natural vibrational frequencies. Resonance shows up in many physical systems and is hence of great importance.

The fact that resonance occurs in the Driven Harmonic Motion Analyzer used in this experiment was made use of, by modeling, to calculate the viscosities of different media.

THEORY:

As shown in Figure 1, a body executing forced oscillations under damping is acted upon by three forces simultaneously: the restoring force -kx, the viscous force b$\dot{x}$, and the external or driving force $F_{forcing} = F_o \cos \omega t$. Applying Newton's second law to the oscillating mass system, we get

$$m\ddot{x} + b\dot{x} + kx = F_o \cos \omega t \qquad (1)$$

Dividing both sides of equation (1) by the mass, m yields

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = A_o \cos \omega t \qquad (2)$$

The solution to equation (3) is of the following form

$$x_{\text{solution}} = x_{\text{complementary solution}} + x_{\text{particular solutior}}$$

The complementary solution is linear combination of sine and cosine terms and is determined by solving the differential equation as a linear homogenous system. The complementary solution to equation (3) is

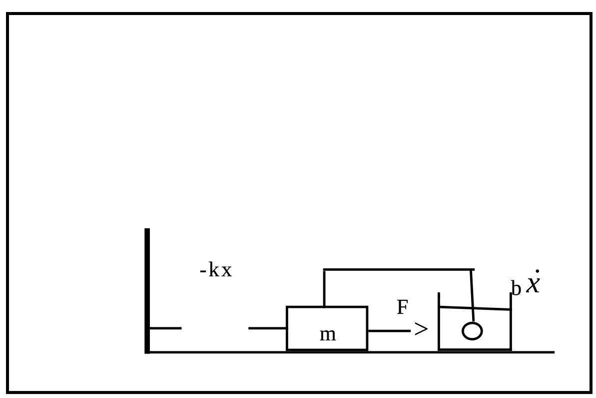

Figure 1: A simplified diagram of a damped harmonic oscillator.

$$x_c(t) = e^{-\beta t}[A_1 \exp(\sqrt{\beta^2 - \omega_o^2}) \quad (3)$$
$$+A_2 \exp(-\sqrt{\beta^2 - \omega_o^2})]$$

and the particular solution for equation (2) is

$$x_p(t) = D\cos(\omega t - \delta) \quad (4)$$

where D is the amplitude of the particular solution, and δ is the phase shift between the driving force and the system response. The complementary solution given by the equation (3), is a short-lived transient: the amplitudes it describes die out due to the exponential decay term. After waiting long enough, only the particular solution, given by equation (4) is worthy of consideration. Substituting equation (4) into equation (3) yields the relation

$$\{A - D[(\omega_o^2 - \omega^2)\cos\delta + 2\omega\beta\sin\delta]\}\cos\omega t \quad (5)$$
$$-\{D[(\omega_o^2 - \omega^2)\sin\delta + 2\omega\beta\cos\delta]\}\sin\omega t = 0$$

Solving for D gives the equation

$$D = \frac{A}{\sqrt{(\omega_o^2 - \omega^2)^2 + 4\omega^2\beta^2}} \quad (6)$$

The resonance frequency can be determined by maximizing the amplitude function with respect to frequency. This is achieved by taking the partial derivative of equation (6) and equating it to zero. The resulting relationship is

$$\omega_R = \omega_o^2 - 2\beta^2 \quad (7)$$

where ω_R is the angular frequency at resonance.
In the case of weak damping, it can be assumed that $\beta << \omega_o$ and $\omega_R \quad \omega_o$, Therefore let $\Delta\omega = (\omega_o - \omega)$ so that

$$(\omega_o^2 - \omega^2) = (\omega_o - \omega)(\omega_o + \omega) \approx -2\omega_o\Delta\omega \quad (8)$$

which when substituted into equation (6) gives

$$D = \frac{A}{\sqrt{4\omega_o^2(\Delta\omega)^2 + 4\omega_o^2\beta^2}} = \frac{A}{2\omega_o^2\sqrt{(\Delta\omega)^2 + \beta^2}} \quad (9)$$

At maximum amplitude of the particular solution, $\Delta\omega = 0$. If this amplitude is called D_{max} then

$$D_{max}(\omega_o = \omega) = \frac{A}{2\omega_o\beta} \quad (10)$$

or

$$\frac{A}{2\omega_o} = \beta D_{max} \quad (11)$$

Combining eqn. (11) with eqn. (9) we get

$$D = \frac{\beta D_{max}}{\sqrt{(\omega_o - \omega)^2 + \beta^2}} \quad (12)$$

which is a Lorentzian line shape.

EXPERIMENTAL SETUP:

The sole apparatus for this experiment was a Pasco Scientific Driven Harmonic Motion Analyzer, Model 9210. The Motion Analyzer unit was leveled using the adjustable screws under its base. A bubble leveling device was used as a guide. It was ensured that there was no friction between the mass bar/displacement scale and the upper mass guide, and also the damping rod and the lower mass guide. This was achieved by slightly rotating the lower and upper mass guides so as to prevent them from rubbing against the displacement scale and damping rod.

EXPERIMENTAL PROCEDURE:

The frequency of the driving force was increased gradually until the amplitude of the mass was noticeable. The smallest amplitude observed was 16mm at a driving frequency of 1.60Hz. From here, the frequency was increased by 0.02Hz and the corresponding amplitudes were recorded. Each time the frequency was increased in this manner, the system was allowed sufficient time (typically 30 s to 1 minute) to stabilize. A digital display on the motion analyzer could be set to display either the driving frequency, or the amplitude, by changing the position of a Function knob.

The Motion Analyzer measures the driving frequency by an optical sensor that counts bars on a rotating disc. As a result, it would not rotate the same number of bars past the sensor every time. The reading on the digital display, indicated for example 1.64Hz on one reading and 1.65Hz on the next. The fluctuating frequency values made it seem difficult to take an accurate measurement of frequency, but the digital display was watched for a few seconds and a mental average was taken. Since the driving frequency knob rotates smoothly and without clicking at specific intervals, it had to be rotated extremely slowly to obtain a difference of 0.02Hz for successive readings.

The amplitude of the mass is measured similarly by an optical sensor counting bars on the Mass bar and displacement scale. Fluctuations in the amplitude were compensated for by taking a mental average of the different (but nearby) values of amplitude.

In the first data set, the mass was allowed to oscillate freely in air. For the second data set , the damping rod was placed into a cylinder containing water. It was made sure that the mass was not rubbing against the sides of the cylindrical vessel while oscillating.

EXPERIMENTAL RESULTS:

While oscillating in air, the amplitude of the mass increased dramatically at resonance. It jumped from 101± 2mm at a frequency of 1.75 ± 0.01Hz to 162 ± 2mm at a resonant frequency of 1.76 ± 0.01 Hz . In water the resonant amplitude was significantly smaller, and the resonant frequency was slightly shifted. The amplitude of the mass in water, at resonance was found to be 81± 2mm at a frequency of 1.78 ± 0.01Hz. The increase in amplitude at resonance was not as dramatic. The amplitude in water just before resonance was 74 ± 2mm at a frequency of 1.76 ± 0.01Hz.

The amplitude of the oscillating mass squared was plotted against the angular frequency for air and water using Igor Pro version 3.01. Figure 2(a) show the plot for the mass oscillating in water. Figure 2(b) shows the same for air.

The graphs obtained are approximately symmetrical about their peaks. This demonstrates that only at a particular frequency, the response of the system is maximum, at progressively higher or lower frequencies the response decreases. The Lorentzian best fit curve fits the data very well. At low amplitudes, the co-ordinates are smeared close together, however at amplitudes close to the resonant amplitude, the systems response increased sharply for a slight increase in frequency, and hence data co-ordinates are farther apart. In Figure 2(b), the amplitudes of oscillation in air and water are almost the same till an angular frequency of 10.8 rad/s, close to the resonant angular frequency there is a substantial difference in amplitude. The amplitudes tend to become equal again at an angular frequency of 12.0 rad/s. The resonant peak for the oscillations in water is slightly shifted to the left of the resonant peak for oscillations in air.

A Lorentzian curve was used to fit the co-ordinates in Figure 2 in order to model the data obtained from the experiment. The curve has the following general form

$$y = k_o + \frac{k_1}{(x - k_2)^2 + k_3} \qquad (13)$$

With k_o made equal to zero, the above equation matches the form of equation (14) squared. Hence we can identify $y = D^2$, $x = \omega$, $k_1 = \beta^2 D^2_{max}$, $k_1 = \beta^2 D^2_{max}$ and $k_3 = \beta^2$.

A graph of the amplitude squared (y-axis) was plotted against the angular frequency (x-axis) using Igor Pro. (version 3.01). Igor also generated the values of y, and k_1 , k_2 and k_3 with their respective uncertainty errors.

The ratio of viscosities is approximately equal to the ratio of damping constants,

$$\frac{\eta_{water}}{\eta_{air}} \approx \frac{\beta_{water}}{\beta_{air}} \qquad (14)$$

The ratio of the viscosity of water to that of air was found by substituting the values of β for air and water (generator by Igor Pro version 3.01) in equation (14).

Igor Pro uses a least squares method in order to formulate a curve to fit the data. It also reports the values of the co-efficients in the Lorentzian function used and their associated error. Table 1 shows these results for air and water.

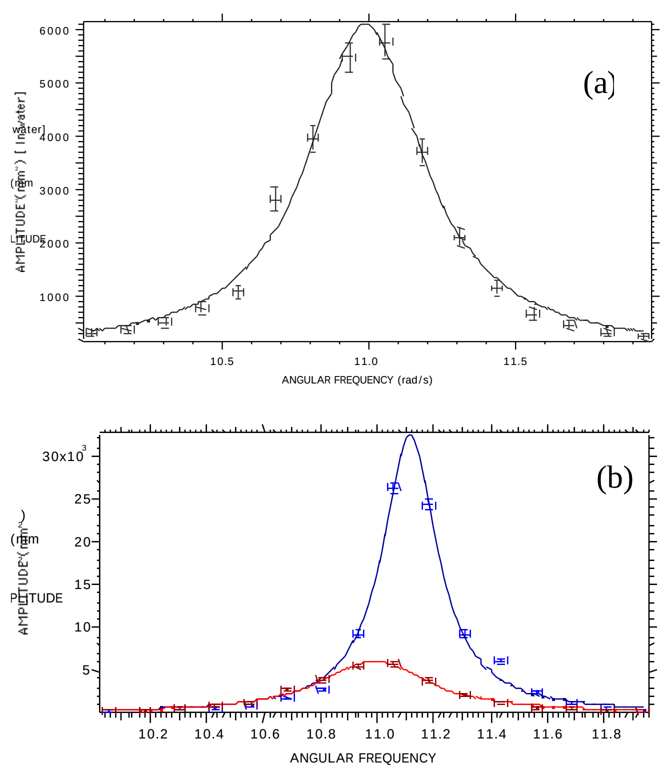

Figure 2: Amplitude squared versus angular frequency for (a) water , and (b) water and air compared. Note the shift in the resonant frequency.

Table 1: Parameter values and their associated errors in air and water.

Parameter	AIR	WATER
$k_o = 0$	0	0
$k_1 = \beta^2 D^2_{max}$	457 ± 34	340 ± 25
$k_2 = \omega_o$	11.119 ± 0.002	10.99 ± 0.01
$k_3 = \beta^2$	0.0141 ± 0.0014	0.0558 ± 0.0051

Table 2: Parameter values of interest and their associated errors.

Parameter	AIR	WATER
β rad/s	0.12 ± 0.006	0.240 ± 0.002
D^2_{max} mm²	$(3.24 \pm 0.41)x10^5$	$(6.09 \pm 0.71)x10^3$
ω_o rad/s	11.119 ± 0.002	10.99 ± 0.01

From equation (14),we know that

$$\frac{\eta_{water}}{\eta_{air}} \approx \frac{\beta_{water}}{\beta_{air}} = 0.24/0.12 = 2.0 \pm 0.05$$

The published values for η_{water} is 1.0 x 10^{-3} (Ns/m^2) and for η_{air} is 1.8 x 10^{-5} . [1]

CONCLUSION:

The value of the ratio of the viscosities of water to air is extremely far off from the published value. This could be a result of the following: The mass was “X” shaped instead of being a circular disc, and the damping rod was extremely thin, due to which the oscillating system did not experience as much resistance due to viscosity as an object otherwise would. The Lorentzian curve fit the data co-ordinates extremely well suggesting the theory used to model the experimental results was appropriate. Also the resonant peaks in water and air were significantly different as predicted by theory. Owing to these reasons it can be concluded that resistance due to viscosity was cut down due to the nature of the construction of the mass. The experiment could be extended to investigate the effect of increased cross sectional area for constant mass, on the ratio of the viscosities of air and water.

[1] Resnick, Robert, Halliday, David and Krane, Kenneth, Physics, 4th Edition, Volume 1. John Wiley and Sons, (1994).