Modeling the viscous torque acting on a rotating object

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By drawing an analogy between linear and rotational dynamics, an equation describing the viscous torque acting on an object rotating in a fluid can be anticipated. Stokes' and Newton's models of a viscous drag force are commonly used to describe the damping force acting on an object moving linearly through a fluid. This experiment demonstrates that these models can be extended to describe the viscous torque that damps the rotation of an object in a fluid. When the rotating object is round, the fluid flow is laminar; hence, the viscous torque is proportional to the angular velocity to the first power, analogous to Stokes' model. However, when the rotating object is rough, causing the fluid flow to be turbulent, the viscous torque is proportional to the angular velocity to the second power, analogous to Newton's model. In addition, this experiment demonstrates that the proportionality constant between viscous torque and angular velocity is dependent on the shape of the object, as is the case in both Stokes' and Newton's models.

INTRODUCTION

As an object moves through a fluid, the viscosity of the fluid acts on the moving object with a force that resists the motion of the object. Two common approaches for modeling this resistive force are the Stokes' and Newton's models. In 1845, Sir George Gabriel Stokes published equations of viscous flow.[1] In particular, Stokes determined the resistive force, or viscous drag force, of a sphere falling under the force of gravity in a fluid, either liquid or gas, to be directly proportional to the sphere's velocity. Stokes model most accurately describes objects moving linearly in a fluid that moves with *laminar* or steady flow.[2]

Sir Isaac Newton, however, showed that the drag force is proportional to the square of the velocity of the object and acts in the directions opposite to the direction of the velocity. Newton's model is associated with higher velocities and *turbulent*, or non-steady, flow.[3] This experiment investigates the relationship between frictional *torque* and *angular velocity*, by drawing an analogy between the linear quantities discussed above and measurable angular quantities.

THEORY

Most commonly, viscous drag is modeled by either Stokes' or Newton's models. Both models agree that the drag force acts in the direction opposite to the velocity vector for the object in motion. The viscous drag force according to Stokes' model for an object in laminar flow is given by:

$$\vec{F}_D = -c_1\vec{v} \qquad (1),$$

where c_1 is a proportionality constant that depends upon the viscosity of the fluid and the shape of the object and v is the instantaneous velocity of the object.[2]

The viscous drag force according to Newton's model for an object in turbulent flow is given by:

$$\vec{F}_D = -c_2 v^2\hat{v} \qquad (2),$$

where c_2 is a proportionality constant that again depends upon the viscosity of the fluid and the shape of the object.

Based on these models, this experiment allows for the resistive drag force to be proportional to some other power of v. Such a drag force would be given by:

$$\vec{F}_D = -c_3 v^n\hat{v} \qquad (3),$$

where c_3 is another proportionality constant depending on the fluid's viscosity and the object's shape, and n is the power of the velocity.

By drawing an analogy between linear and rotational dynamics, the rotational counterpart to such a drag force can be anticipated. In particular, by substituting the corresponding rotational quantities into equation (3), the frictional torque acting on an object rotating with angular velocity ω might be modeled by:

$$\tau_D = I\alpha = I\frac{d\omega}{dt} = -c_4\omega^n \qquad (4).$$

Then solving equation (4) for dω/dt, yields:

$$\frac{d\omega}{dt} = -\frac{c_4}{I}\omega^n = -k\omega^n \qquad (5),$$

where $k = c_4/I$ is a "new" proportionality constant. Differential equation (5) can easily be solved for various values of the exponent n. For instance, if n = 1, equation (5) can be solved using the separation of variables technique to yield:

$$\omega = \omega_o e^{-kt} \qquad (6),$$

where ω_o is the initial angular velocity. And, if n = 2, the solution to equation (5) is given by:

$$\frac{1}{\omega} = kt + \frac{1}{\omega_o} \qquad (7).$$

EXPERIMENT

This experiment uses an Ealing air gyroscope (#13-2209). The steel rotor ball of the gyroscope is the rotating mass that is studied. The angular velocity, or frequency of rotation, of the rotor ball is determined using a laser. Black strips of electrical tape are placed on the upper portion of the shiny rotor, and a laser is aligned so that it reflects off this portion of the rotor. Then the laser beam is focused through a lens onto the tiny pinpoint head of a fast photodiode. Figure (1) below illustrates the circuitry setup for the photodiode.

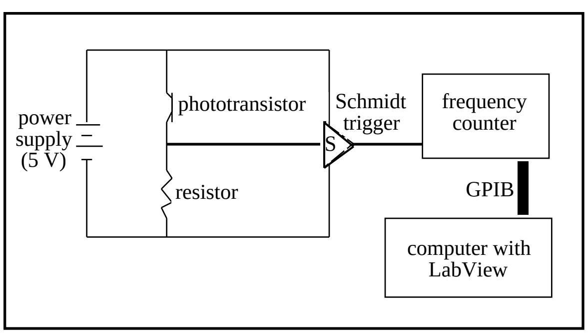

Figure (1). Circuitry involved with the photodiode.

When the laser light reflects onto the photodiode (or phototransistor), the current supplied by the power supply easily passes through the phototransistor, causing a high voltage reading to pass through the Schmidt trigger. But, when no light is incident on the phototransistor, little current makes it to the resistor, sending a lower voltage reading to the Schmidt trigger. The Schmidt trigger then shapes the voltage signals into a sharp square wave, which can then be read by a Hewlet Packard frequency counter (5385A). In addition, the frequency counter is connected to a Macintosh computer with a GPIB cable, in order for a LabView algorithm to import the data acquired through the frequency counter.

Finally, a tank of compressed nitrogen gas is connected by a tube to the air gyroscope apparatus. The pressure of the N_2 gas coming out of the tank is adjusted to a constant 12 psi throughout the experiment, so that the rotor "sits" on a cushion of gas.

A LabView (v.3.1) program "Viscous Torque.LV3," was written to set the controls for and to import data from the frequency counter. The rotor is spun by hand to attain the highest possible initial angular velocity. Then, as the rotor spins, the LabView program calculates the average ω value over 10.0 second intervals and records a table of values for ω and the corresponding time to a file designated by the user.

In order to more thoroughly investigate the relationship between frictional torque and angular velocity, the rotor is altered by adding surface area to the rod, which causes the rotation to be damped more quickly. Rectangular pieces of styrofoam board, each measuring 0.5 cm thick by 3.0 cm wide by 6.0 cm long, are used for the additional area. In order to help keep the rod upright as it rotates, two rectangular pieces are added opposite one another (like wings spanning approximately 12 cm) to the rod of the rotor for each run. Each pair of styrofoam pieces is referred to as A, so that the run with no additional area is run 0A, with two additional pieces is run 1A, with four additional pieces is run 2A, and with six additional pieces is run 3A.

ANALYSIS AND INTERPRETATION

In order to determine if any of the data can be modeled by equation (6), a semi-log plot of ω versus time for all data is performed.

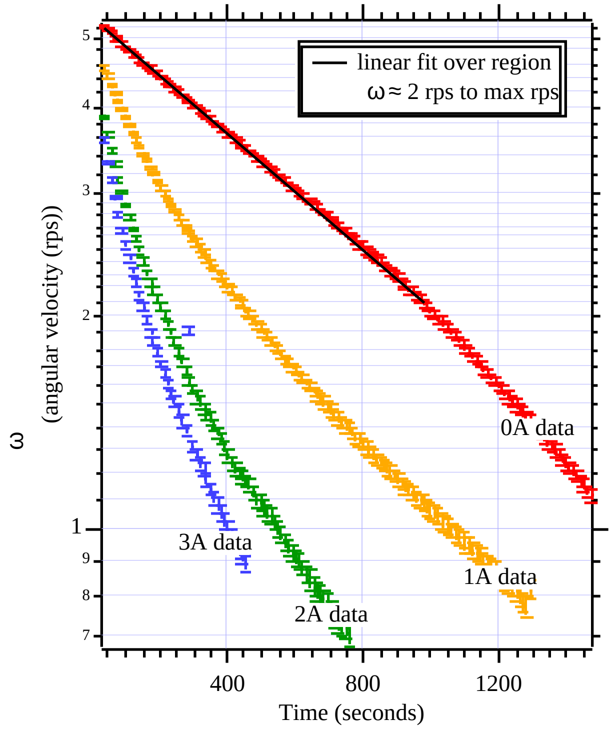

Figure (2). Semi-log plot of ω versus time for all data sets, where a linear result implies that the exponent n = 1 in equation (4).

Since only the 0A data in figure (2) appears to have a straight, linear result on the semi-log plot, only the 0A data, or data taken without additional area, is accurately modeled by equation (4) with n = 1.

In order to determine the value of n for the other sets of data, equation (5) is considered. Thus, dω/dt versus ω is plotted, using Igor Pro (v.3.0). A powerlaw fit is performed to determine the values of k and n for each data set. In particular, we are interested in fitting a function of the form:

$$y = c_1(x)^{c_2} \qquad (9),$$

The constant c_1 represents k, and c_2 represents n from equation (5). Thus, on a log-log plot, the slope of a linear fit gives the exponent n and the intercept relates to the proportionality constant k, such that log(k) is the intercept.

Table (1) below lists the outcomes of the powerlaw fit, and figure (3) shows the log-log plot of dω/dt versus ω for all four sets of data. All of the data is fit over the more accurate region of higher ω values, since at smaller ω values all sources of error have a greater impact. Specifically, the data is fit from the maximum ω, or initial ω_o, to approximately ω = 2 rps.

Table (1). k and n values (from equation (5)) for all four data sets, including standard deviation based on the powerlaw fit.

data	slope = n	intercept = k
0A	1.03±0.04	$(9.3\pm0.5)\times10^{-4}$
1A	1.92±0.02	$(7.0\pm0.2)\times10^{-4}$
2A	2.04±0.05	$(13.0\pm0.9)\times10^{-4}$
3A	2.04±0.08	$(17.2\pm1.4)\times10^{-4}$

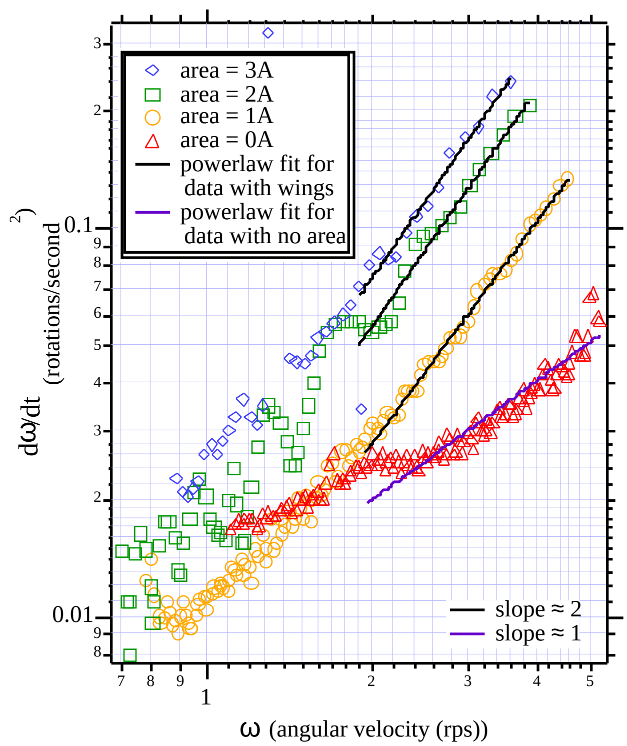

Figure (3). Log-log plot of dω/dt versus ω for all data, where the slope gives the exponent n and the intercept gives the proportionality constant k from equation (5).

As shown in figure (3), the data for 0A has a different slope, or n value, than does the data for 1A, 2A, and 3A. Indeed, in table (1), we see that n is approximately 1 for the data with no additional area, as expected. Also, as shown in table (1), n is approximately 2 for all of the data taken for the rotor with "wings," or additional area. Thus, the viscous torque is directly proportional to angular velocity for the rotor and smooth rod, i.e. τ_D ω^1 for the 0A data, and the viscous torque is proportional to the angular velocity squared for the rotor with wings, i.e. τ_D ω^2 for the 1A, 2A, and 3A data.

Since the 1A, 2A, and 3A data are all modeled by the same equation, analysis of this data leads to an understanding of any dependence in the proportionality constant k on the known physical system. Thus, considering equation (7), the proportional relationship between the torque and the angular velocity squared can be demonstrated. Consequently, a direct plot of the inverse angular velocity versus time will have a linear result for the data with wings, given $\tau_D \quad \omega^2$ for the 1A, 2A, and 3A data. Figure (4) below illustrates this relationship. As in figure (3), a linear fit is shown only for the region from initial ω_o to $\omega \approx 2$ rps, where the data is more accurate.

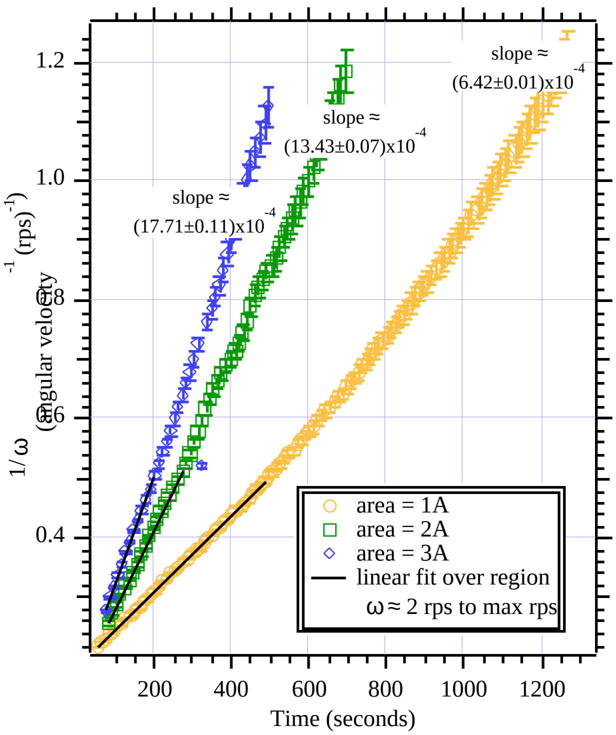

Figure (4). Plot of 1/ω (angular velocity^{-1}) versus time for data with additional area, demonstrating the proportional relationship between τ_D and ω^2 for this data.

The slopes of the linear fits in figure (4) give the proportionality constants k from equation (7). Thus, we note that these slopes as well as the k values listed in table (1) are consistent. In addition, the proportionality constant k appears to increase as the amount of area added to the rod is increased, indicating that the proportionality constant is directly related to the shape of the rotating object.

CONCLUSION

Data analysis has shown that the shape of a rotating object has an interesting impact on the viscous torque. For the smooth rod and rotor, the viscous torque is best modeled by an equation analogous to Stokes' model of viscous force for objects moving in *laminar* flow. This result should be expected, since the air flow around the rod as the rotor rotates will be steady and circular. On the other hand, when "wings" are attached to the rod, the viscous torque is best modeled by an equation analogous to Newton's model of viscous force for objects moving in *turbulent* flow. Turbulent flow is generally associated with vortices forming behind the object as it moves. And, indeed, it is plausible that vortices are created behind the wings attached to the rod, as the rotor spins.

In addition, figure (4) reveals the fact that the proportionality constant between the viscous torque and the square of the angular velocity depends upon the amount of surface area added to the rod. This is also expected, since in both Stokes' and Newton's models the proportionality constant depends upon the shape of the object and the viscosity of the fluid. In this experiment, the viscosity of the fluid(s), a combination of N_2 gas and air, remains constant, except for variations due to temperature and pressure; thus, variations in the proportionality constant are expected to be related only to the shape, or geometry, of the object. And, as additional wings are added to the rod, the shape of the rotating object is effectively changed.

[1]Robert E. Street, *Grolier Encyclopedia* (Grolier Electronic Publishing, Inc., 1995).

[2]Dwight E. Gray, Ph.D. (coordinating editor), *American Institute of Physics Handbook* (McGraw-Hill Book Company, Inc., The Maple Press Company, York, PA, 1957); pp. 2-13.

[3]Joseph Norwood, Jr., *Intermediate Classical Mechanics* (Prentice-Hall, Inc., Englewood Cliffs, NJ, 1979); pp. 52-54.