Properties of Traversable Wormholes in Spacetime

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In this project, the Morris-Thorne metric of the spacetime around a wormhole was investigated to study the properties of the wormhole and determine if the wormhole might be traversable for human travellers. First, a 2-D slice of the metric was took by fixing two coordinates t and θ, and an embedding diagram of the wormhole surface was created. The diagram showed that the wormhole connects two asymptotically flat spaces. Then, a plot of the geodesic was made to visualize how a free-falling particle travels through the wormhole. Also, it was concluded that a traveller experiences small tidal force during the trip if the velocity is small, make the trip through the wormhole possible. Finally, the stress-energy-momentum tensor showed that the spacetime for the Morris-Thorne wormhole requires negative energy density, which is only allowed on a microscopic level for a short time.

I. INTRODUCTION

A wormhole is an alternate path that allows particles travel from one point to another. which distinct from the usual path in flat spacetime. Wormholes are considered to be a new hope for interstellar travel, since the path through the wormhole has a smaller time interval, and, most importantly, they are mathematically constructed solutions to Einstein's equations for gravity. In 1988, Michael Morris and Kip Thorne introduced a simple wormhole solution [**?**] to the Einstein's equation, which is easy to visualize. In this paper, I will construct the properties of the wormhole base on the metric, and make conclusion about a particle travelling through it.

II. SPACETIME AND GENERAL RELATIVITY

A. Coordinates

For classical 3-dimensional coordinates, we use the sets of basis $\{r, \theta, \phi\}$ or $\{x, y, z\}$, which represents only spatial dimensions. For the 4-dimensional spacetime coordinates, we introduce an extra time dimension ct, where t is the time measured by a static observer and the constant c is the speed of light.

The four coordinates can be expressed using the notation x^μ, which is a collection of four variables with indices μ goes from 0 to 3. Each variable represents one dimension in the coordinate system by

$$x^\mu \leftrightarrow \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} ct \\ r \\ \theta \\ \phi \end{bmatrix}. \tag{1}$$

B. The Metric

In mathematics, the metric $d : M \times M \to [0, \infty)$ is a function that defines the distance between a pair of points in a metric space M. In physics, we usually call it the spacetime interval. Suppose we have a 4-D vector space V with basis $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Let $d\mathbf{q}$ be an infinitesimal displacement, Then the vector $d\mathbf{q}$ can be expressed in the form

$$d\mathbf{q} = \sum_{\mu=0}^{3} dx^\mu \, \mathbf{e}_\mu. \tag{2}$$

On the right side of Eq. (2), the dx^μ is the vector component on the basis $\mathbf{e}_\mu$. Now we find the line element ds by taking the dot product of the vector dx^μ on itself. We have

$$ds^2 = \langle d\mathbf{q}, d\mathbf{q} \rangle \tag{3}$$

$$= \left(\sum_{\mu=0}^{3} dx^\mu \, \mathbf{e}_\mu \right) \cdot \left(\sum_{\nu=0}^{3} dx^\nu \, \mathbf{e}_\nu \right) \tag{4}$$

$$= \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} (\mathbf{e}_\mu \cdot \mathbf{e}_\nu) \; dx^\mu dx^\nu. \tag{5}$$

$$\tag{6}$$

We define a 4×4 matrix with elements $g_{\mu\nu} = \mathbf{e}_\mu \cdot \mathbf{e}_\nu$. Then

$$ds^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu. \tag{7}$$

In short, we write

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{8}$$

Notice that there are the same indices μ, ν put at the top and the bottom. That implies a sum over indices μ, ν from 0 to 3.

We use the notation $g_{\mu\nu}$ to represent the rank-2 metric tensor. In a 4-dimensional spacetime, the indices μ, ν goes from 0 to 3. Then the metric has the form of a 4×4 matrix

$$g_{\mu\nu} \leftrightarrow \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}. \tag{9}$$

For example, in a 3-D Euclidean space $\mathbb{R}^3$ with the set of orthonormal basis $\{\hat{x}, \hat{y}, \hat{z}\}$, we have

$$\mathbf{e}_\mu \cdot \mathbf{e}_\nu = \delta_{\mu\nu} = \begin{cases} 1 & \mu = \nu, \\ 0 & \mu \neq \nu. \end{cases} \tag{10}$$

The the metric tensor has the form of a 3×3 matrix

$$g_{\mu\nu} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{11}$$

and the inverse of the metric tensor $g_{\mu\nu}$ is denoted by $g^{\mu\nu}$. Then we have

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu && (12) \\ &= \begin{bmatrix} dx & dy & dz \end{bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} && (13) \\ &= dx^2 + dy^2 + dz^2, && (14) \end{aligned}$$

which is the space interval on a 3-D Euclidean space. Also, if we use spherical coordinates $\{r, \theta, \phi\}$ as the set of orthogonal basis. Then the metric has the form of a 3×3 matrix

$$g_{\mu\nu} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}. \tag{15}$$

Then the metric takes the form

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2. \tag{16}$$

From the above examples, notice that even in the same space, the metric has different forms depending on the basis we use. If we define the vector $dx_\mu = g_{\mu\nu} dx^\nu$, Eq. (8) becomes

$$ds^2 = dx_\mu dx^\mu, \tag{17}$$

The vector dx^μ is the contravariant component, with the index μ on top. Thus we define dx_μ to be its covariant counterpart, with the index μ on the bottom. The scalar product of the two is invariant under coordinate transformation, even Lorentz transformation. As a result, the dot products in Eq. (14) and (16) are equivalent, even with different basis. This is crucial since observers in any inertial frame should agree on the spacetime interval.

III. MORRIS-THORNE-WORMHOLE

In the above section, we have developed some mathematical definitions and notations. Then we can use them to study the properties of wormholes.

A. The Metric

The metric introduced by Machael Morris and Kip Thorne [?] is described by

$$ds^2 = -c^2 dt^2 + dl^2 + (b_0^2 + l^2)(d\theta^2 + \sin^2\theta \, d\phi^2), \tag{18}$$

where:

- t is the time measured by a static observer, and $-\infty < t < \infty$.
- θ, ϕ are spherical polar coordinates, and $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$
- l is the radial coordinate, and $-\infty < l < +\infty$.

For $l^2 \gg b_0^2$, the metric takes the form of Eq. (16), which is a representation of flat spacetime in spherical coordinates. It shows that the spacetime has two asymptotically flat regions at the limits of $l \to +\infty$ and $l \to -\infty$.

B. Embedding Diagram

The metric describes the structure of the wormhole in 4 dimensions. To visualize the wormhole, we first take a 2-D slice of the metric by fixing $t = t_0$ and $\theta = \pi/2$. Then if we let $l = \sqrt{r^2 - b_0^2}$, where r is the radial coordinate used in polar coordinates, the metric become

$$ds^2 = \left(\frac{r^2}{r^2 - b_0} \right) dr^2 + r^2 d\phi^2. \tag{19}$$

Now we have a curved 2-dimensional surface in polar coordinates (r, ϕ). We can embed the surface in a 3-dimensional space by expressing it in cylindrical coordinates (r, ϕ, z), with

$$\begin{aligned} ds^2 &= dr^2 + r^2 d\phi^2 + dz^2 && (20) \\ &= \left(1 + \left(\frac{dz}{dr} \right)^2 \right) dr^2 + r^2 d\phi^2. && (21) \end{aligned}$$

We solve for z using Eqs. (19) and (21), and we get

$$z = \pm\, b_0 \ln \left(r + \sqrt{\left(\frac{r}{b_0} \right)^2 - 1} \right). \tag{22}$$

Then we can plot a slice of the surface of the wormhole is a 3-D diagram.

Figure 1 shows the curved 2-D surface of the wormhole being embedding in a 3-D space. For the metric to work, we assume that the wormhole connects two asymptotically flat regions, the spacetime gets flatter and flatter as $|l|$ increases and $l^2 \gg b_0^2$ by comparison. It also shows that wormhole has a throat where $l = 0$ and $r = b_0$.

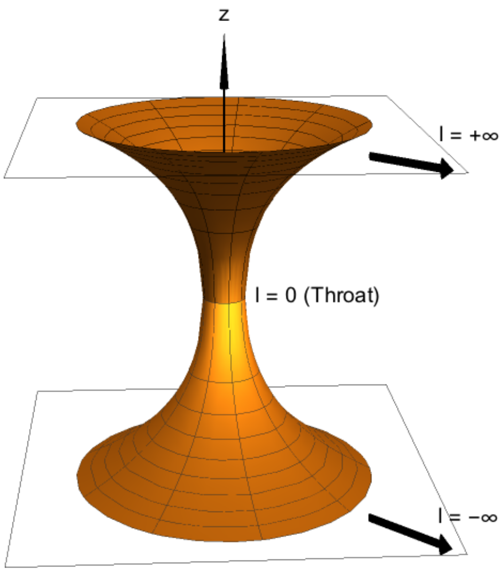

FIG. 1: The embedding diagram of the Morris-Thorne wormhole in cylindrical coordinates. Fixed $t = t_0$ and $\theta = \pi/2$. There are also 2 asymptotically flat regions on top and bottom.

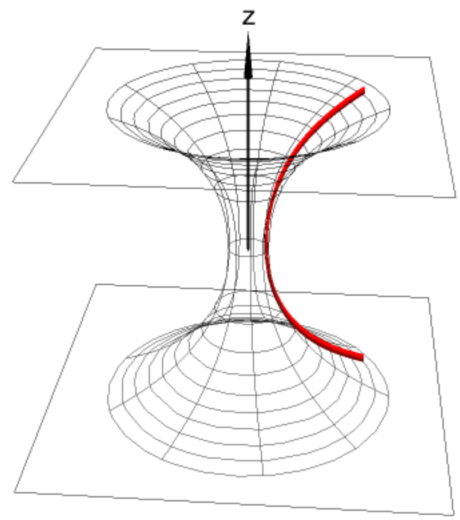

FIG. 2: The geodesic (red line) of the free-falling particle on the surface of the wormhole.

C. Geodesic

There is a great deal of information we can get from the metric. A geodesic is the path of a free-falling particle, with no external force except gravity acting on it, moving along the space. First we can find the connection coefficient Γ from the metric by

$$\Gamma^{\mu}_{\nu\alpha} = \frac{1}{2} g^{\mu\beta} \left(\frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} + \frac{\partial g_{\beta\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\nu\alpha}}{\partial x^{\beta}} \right). \tag{23}$$

The connection coefficient gives us information about how the tangent vector changes as it moved along the space. In a curved space, if we parallel transport a tangent vector A_μ for an infinitesimal displacement along dx^β, the vector components of A^μ generally changes. Suppose the new tangent vector is $A^{\mu'} = A^\mu + \delta A^\mu$ in the parallel transport. Then

$$\delta A^\mu = -\Gamma^{\mu}_{\nu\beta} A^\nu dx^\beta. \tag{24}$$

Note that $\Gamma^{\mu}_{\nu\beta}$ determines the change of the vector components in parallel transport.

The geodesic equation is

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^{\mu}_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0. \tag{25}$$

The variable τ is a scalar parameter. For massive particles, τ can be thought of the proper time. Notice the first term on the left side of Eq. (25) is the acceleration in respect to each coordinate, and the second term of Eq. (25) tells us how the path is curved due to curvature of the space itself. At the limit of a flat spacetime where $\Gamma^{\mu}_{\nu\alpha} = 0$. Then Eq. (25) becomes

$$\frac{d^2 x}{dt^2} = \frac{d^2 y}{dt^2} = \frac{d^2 z}{dt^2} = 0, \tag{26}$$

which agrees with Newton's First law for a body travelling straight with no net force acting on it.

If we use the connection coefficient for the Morris-Thorne wormhole, we find that the particle moves along the world line

$$l = vt, \tag{27}$$

$$\theta = \text{const}, \tag{28}$$

$$\phi = \text{const}, \tag{29}$$

where v is the velocity of the particle. The proper time interval $\Delta\tau$ for travelling from one point to another is given by

$$\Delta\tau = \int \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} d\tau. \tag{30}$$

Suppose there is a particle free-falling through the wormhole with velocity $0.1c$, and it travels from the point $\tau_1 = -10$ to the point $\tau_2 = 10$. Then $c\,\Delta\tau = 19.9$ meters. Thus the proper time interval is $\Delta\tau = 6.64 \times 10^{-8}$ second, which is nearly instantaneous. In theory, a wormhole can directly connect two points that are arbitrary far apart, but the trip only takes small amount of time. That shows the path through the wormhole has a much smaller the time interval.

Figure 2 shows the geodesic of the free-falling particle expressed in cylindrical coordinates. The plot was overlapped with the embedding diagram of the wormhole, showing how the particle travels along the surface of the wormhole.

D. Tidal Force

When free-falling through the wormhole, we expect a tidal force that stretches and twists the traveller. That means a traveller will not survive the trip if there is strong tidal force. To understand the tidal acceleration on the free-falling particle, we have to consider the Riemann curvature tensor in the frame of the particle.

Then we can find the Riemann curvature tensor by

$$R^{\mu}_{\nu\alpha\beta} = \Gamma^{\mu}_{\nu\beta,\alpha} - \Gamma^{\mu}_{\nu\alpha,\beta} + \Gamma^{\gamma}_{\nu\beta}\Gamma^{\mu}_{\gamma\alpha} - \Gamma^{\gamma}_{\nu\alpha}\Gamma^{\mu}_{\gamma\beta} \tag{31}$$

However, all the calculations above took place in the reference frame of a static observer, which is the unbarred frame. Now we need information in the frame of the free-falling particle, the barred frame. Then we need to apply Lorentz transformation to the tensor. Suppose the particle is free-falling with velocity v, then the Lorentz-transform matrix has the form

$$L^{\mu}{}_{\bar{\nu}} \leftrightarrow \begin{pmatrix} \gamma & \gamma\frac{v}{c} & 0 & 0 \\ \gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{32}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$. Then we obtain the Riemann curvature tensor $R_{\bar{\mu}\bar{\nu}\bar{\alpha}\bar{\beta}}$ in the frame of the free-falling particle by the set of equations

$$R_{\delta\lambda\rho\sigma} = g_{\delta\gamma} R^{\gamma}_{\lambda\rho\sigma}, \tag{33}$$

$$R_{\bar{\mu}\bar{\nu}\bar{\alpha}\bar{\beta}} = L^{\delta}{}_{\bar{\mu}}\, L^{\lambda}{}_{\bar{\nu}}\, L^{\rho}{}_{\bar{\alpha}}\, L^{\sigma}{}_{\bar{\beta}}\, R_{\delta\lambda\rho\sigma}, \tag{34}$$

where $R_{\delta\lambda\rho\sigma}$ and $R^{\gamma}_{\lambda\rho\sigma}$ are simply two different forms of the Riemann curvature tensor in the frame of the static observer. Using Eq. (34) with some calculation, it was found that

$$R_{\bar{2}\bar{0}\bar{2}\bar{0}} = R_{\bar{3}\bar{0}\bar{3}\bar{0}} = -\left(\frac{\gamma v}{c}\right)^2 \frac{b_0^2}{(b_0^2 + l^2)^2}. \tag{35}$$

Also for $\mu, \nu \neq 2$ and $\mu, \nu \neq 3$,

$$R_{\bar{\mu}\bar{0}\bar{\nu}\bar{0}} = 0. \tag{36}$$

Then the tidal acceleration vector $A^{\bar{\mu}}$ can be found by

$$A^{\bar{\mu}} = -g^{\bar{\mu}\bar{\alpha}} R_{\bar{\alpha}\bar{0}\bar{\nu}\bar{0}} x^{\bar{\nu}}, \tag{37}$$

where the barred indices indicate that the calculation takes place in the barred frame. We can conclude from Eq. (35) and (37) that at the limit of $v \to 0$, the tidal acceleration also goes to zero and vanishes. That means theoretically, a traveller with small velocity can survive the trip through the wormhole, making the wormhole traversable.

IV. STRESS-ENERGY-MOMENTUM TENSOR

In 4-dimensional space-time, the classical momentum $\mathbf{p}$ of a massive particle is generalized to the 4-momentum p^{μ} by

$$p^{\mu} \leftrightarrow \begin{bmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{bmatrix} = \begin{bmatrix} E/c \\ p_x \\ p_y \\ p_z \end{bmatrix}, \tag{38}$$

where E is the relativistic energy and $\mathbf{p} = (p_x, p_y, p_z)$ is the spatial momentum.

The stress-energy-momentum tensor $T_{\hat{\mu}\hat{\nu}}$ is defined to be [**?**]

$$T_{\mu\nu} \leftrightarrow \frac{dp^{\mu}}{d^3V_{\nu}}, \tag{39}$$

which means 4-momentum per 3-volume. We specifcally look at $T^{\hat{0}\hat{0}} = \dfrac{dE}{dx\,dy\,dz}$, which is called the energy density ρ. The indices $\hat{\mu}, \hat{\nu}$ indicates that the tensors or vectors are normalized.

Since $R^{\mu}_{\nu\alpha\beta}$ is a rank-4 tensor, we can contract it to a rank-2 tensor. For example,

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} \tag{40}$$

and

$$R^{\mu}_{\nu} = g^{\mu\alpha} R_{\alpha\nu}, \tag{41}$$

where $g^{\mu\nu}$ is the inverse of the metric tensor $g_{\mu\nu}$. Then the scalar curvature R of the spacetime surface can be obtained by

$$R = R^{\mu}_{\mu}. \tag{42}$$

Finally, the Einstein curvature tensor $G_{\mu\nu}$ is defined to be

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}. \tag{43}$$

Then Einstein's equations relate the Einstein curvature tensor $G_{\mu\nu}$ and the stress-energy tensor $T_{\mu\nu}$ by

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{44}$$

which is a proportionality relationship. Note that $G_{\mu\nu}$ is about the curvature of the space and $T_{\mu\nu}$ is about the mass and energy present in the space. Then Einstein's equations suggest that spacetime is curved due to the presence of mass and energy, and then the motion of bodies depends on the geodesic of the curved spacetime. This is a generalization of gravity between bodies.

In most cases, we work with the contravariant form of the stress-energy-momentum tensor, denoted by $T^{\mu\nu}$, which is given by

$$T^{\mu\nu} = g^{\mu\alpha} g^{\mu\beta} T_{\alpha\beta}. \tag{45}$$

Then, to normalize the tensors, we have

$$T^{\hat{\mu}\hat{\nu}} = \sqrt{|g_{\mu\mu}g_{\nu\nu}|}\,T^{\mu\nu}. \tag{46}$$

Note that there is no implied sum in Eq. (46). The equation only involves simple multiplication of elements.

The energy density ρ of the wormhole from the view of the static observer can be found using the stress-energy tensor by

$$\rho = T^{\hat{0}\hat{0}} = -\frac{c^4}{8\pi G}\frac{b_0^2}{(b_0^2 + l^2)^2} < 0, \tag{47}$$

which has a negative value. From the above result, an observer near the Morris-Thorne-wormhole will observe negative energy density, which is unlikely to happen on a macroscopic level. However, the quantum field theory does allow negative energy density on a microscopic level for a short time.

V. CONCLUSION

A wormhole is expressed in 4-D spacetime. However, we could take a 2-D slice of the wormhole and visualize it on a 3-D space, which is called the embedding diagram. We could see from the diagram that this metric agrees with the assumption that the wormhole connects two asymptotically flat surface. For a free-falling particle, the geodesic plot also showed how a free-falling particle moves on the surface of the wormhole. A calculation of the proper time interval showed that travelling through a wormhole only takes a small amount of time, which shows how beneficial wormholes are. When the particle is travelling with small velocity, the tidal force on a particle becomes very small That means theoretically a human can travel through the wormhole safely. However, the calculation also showed that an observer near the wormhole will observe negative energy density, which is very unlikely to happen, except in a microscopic level for a short time. That also explains why we have never observed a wormhole anywhere in space.

[1] Larry C Markley and John F Lindner. Artificial Gravity Field. *Results in Physics*, 3:24–29, 2013.

[2] Michael S Morris, Kip S Thorne, and Ulvi Yurtsever. Wormholes, Time Machines, and the Weak Energy Condition. *Physical Review Letters*, 61(13):1446, 1988.