Impact of the Gravitational Force on Star Formation

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In this experiment we examined the impact of the gravitational force on star formation. We changed the explicit radial dependence of the gravitational force and calculated the minimum mass required for a gas cloud to form a star using Mathematica. We then varied the gravitational constant in order to have the minimum mass required under our modified gravitational forces match the minimum mass required under the traditional force law.

I. INTRODUCTION

The motivation for this paper extends from the question that physicists of the future may have to ask one day. Suppose that we live in a time where all of the laws of physics are specified completely and exactly; that is we know everything from the fundamental particles, the various forces, how to perfectly marry quantum mechanics and relativity, etc. What do physicists do next? The interesting thing about this question is the fact that we can ask this question to a degree now. In this paper we examine the impact of changing the structure of Newton's law of universal gravitation on the process of star formation.

II. THEORY

In order to examine the impact of changing the gravitational force on star formation we first derive a generalized form of the law of universal gravitation to allow for a changing number of dimensions. Then we derive an expression for a the Jean's Mass, a factor that limits the possibility of star formation.

A. Gravitational Force Derivation

Newton's law of universal gravitation states that the gravitational force between two point masses m_1 and m_2 separated by a distance r is given by

$$F = G\frac{m_1 m_2}{r^2}, \tag{1}$$

where $G = 6.673 \cdot 10^{-11}$ $\text{m}^3\text{kg}^{-1}\text{s}^{-2}$. However, this law is hiding the true nature of forces from point sources. We can think of a point source as generating a field that is diluted over the surface area of a sphere, $4\pi r^2$. Using this quantity as our denominator for the force, we need to redefine the gravitational constant to be Newton's G multiplied by 4π, $\mathcal{G} = 8.37 \cdot 10^{-10}$ $\text{m}^3\text{kg}^{-1}\text{s}^{-2}$. Now we can write a new expression for our force

$$F = \mathcal{G}\frac{m_1 m_2}{4\pi r^2}, \tag{2}$$

In order to generalize the force of gravity to other numbers of dimensions, it will be helpful to generalize spheres

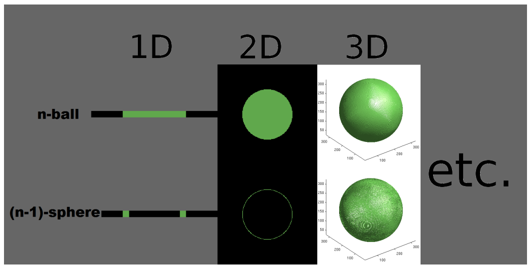

FIG. 1: Schematic showing the (n-1)-sphere and the n-ball for n=1, 2, and 3. Taken from [3].

to other numbers of dimensions. The **(n-1)-sphere**, is the set of all points equidistant from a center point in $\mathbb{R}^n$ [2]. From this definition, the shape commonly referred to as a circle is a 1-sphere and the object commonly referred to as a "sphere" is actually a 2-sphere as shown in Fig. 1. It is important to note that (n-1)-sphere is the surface formed by the points equidistant from the center, not the region bounded by that surface; the 2-sphere is the skin on the basketball, not the air filling it.

The concepts of circumference for the circle and the surface area for the common sphere may be generalized to the surface volume of the (n-1)-sphere. The surface volume of an (n-1)-sphere with radius r is given by

$$S_{n-1} = \frac{n\pi^{n/2}}{\Gamma\left[\frac{n}{2}+1\right] r^{n-1}}, \tag{3}$$

where Γ is the gamma function which generalizes the factorial function to a complex domain by the relationship $\Gamma[n] = (n-1)!$ [2]. The gamma function is used here because the domain of the factorial function is the non-negative integers and we need to allow for both negative and non-integer numbers. By substituting $n-1=1$ we recover the circumference of a circle, $S_1 = 2\pi r$, and by substituting $n-1=2$ we recover the surface area of a common sphere $S_2 = 4\pi r^2$.

We can now generalize this argument to be valid in any number of dimensions n, since the force of gravity will always be proportional to the product of the masses and inversely proportional to S_{n-1},

$$F_n = \mathcal{G}_n \frac{m_1 m_2}{S_{n-1}} = \mathcal{G}_n \frac{m_1 m_2 \Gamma\left[\frac{n}{2}+1\right]}{n\pi^{n/2} r^{n-1}}. \tag{4}$$

At this point I will emphasize that our problem exists in three spatial dimensions. The strength of the field due to a point source is diluted over the surface area of the 2-sphere. This is the crux of the $1/r^2$ force laws that we have in everyday physics. In our argument we are considering a force law where the field due to a point source is diluted over the surface volume of a higher dimensional sphere. We are not considering the formation of stars in dimensions other than three; we are examining how stars would form in three dimensions if we borrowed a force law from a universe with a different of spatial dimensions.

B. Star Formation

Our goal is to investigate how altering the structure of the gravitational force law impacted the process of star formation. In order to do so, we break down the process of star formation into its key steps. We begin by treating a cloud of molecular hydrogen H_2 as an ideal gas. We then consider the fluid dynamics of a coalescing cloud of gas, which allow us to fully specify the gravitational collapse of the gas cloud into a star. The derivation in this section follows closely to the derivation shown in [1].

1. *Equation of State*

In order to begin our journey from a cloud of molecular hydrogen gas to a star we will start by considering the gaseous nature of the hydrogen cloud. We can make the assumption that a cloud of molecular hydrogen gas will behave as an ideal gas. The ideal gas assumption is a valid approximation in this problem since the gas cloud is in a low pressure environment. The ideal gas law relates several key physical quantities of an ideal gas,

$$PV = NkT, \tag{5}$$

where P is the pressure the gas is under, V is the volume of the space that the gas is occupying, N is the total number of molecules in the gas, k is Boltzmann's constant, and T is the temperature of the gas, which we assume to be roughly constant in this case. We can also consider the mass density of the gas cloud, which is given by the mass of the cloud divided by its volume,

$$\rho = \frac{N\mu m_H}{V}, \tag{6}$$

where m_H is the mass of a hydrogen atom and μ is the molecular mass of the gas cloud. Since our gas cloud consists entirely of H_2 molecules, $\mu = 2$. By substituting Eqn. 6 into Eqn. 5 and simplifying we obtain a form of the equation of state,

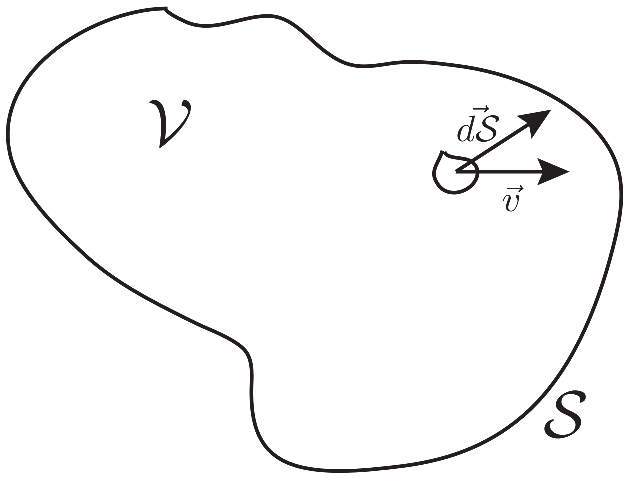

FIG. 2: Arbitrary volume bounded by a surface

$$\frac{P}{\rho} = \frac{kT}{\mu m_H} \equiv a_0^2. \tag{7}$$

If we assume that the temperature of the gas cloud is constant, then the ratio of pressure to density of a gas cloud is equal to the constant a_0^2.

2. *Calculus of Fluid Dynamics*

Given the fact that the molecules in the gas cloud are always moving, we must consider equations to model their motion in order to get an accurate description of the mechanics of the cloud. In general there are two ways to examine the flow of molecules within a gas cloud. The first is to consider a particular point in space and calculate the changes in the gas over time at that particular point. This process is called the Eulerian method. In contrast, the Lagrangian method involves choosing a reference frame where a particular gas particle is fixed. Then we calculate the properties of the gas cloud over time in this reference frame. In this section we develop both Eulerian and Lagrangian time derivatives of the density of the cloud. We will later use the Lagrangian derivative in our calculations for gravitational collapse.

Consider an volume of gas $\mathcal{V}$ bounded by a surface $\mathcal{S}$ as shown in Fig. 2. The shape of the volume does not matter, but the volume must be bounded. Describing the flow of mass through the entire surface at once would be hard, but on an infinitesimal piece of the surface $d\mathcal{S}$, the flow of mass through that region is given by

$$\rho\vec{v} \cdot d\vec{\mathcal{S}}, \tag{8}$$

where $\vec{v}$ is the velocity of the mass moving through $d\mathcal{S}$ and $d\vec{\mathcal{S}}$ is the vector normal to the infinitesimal surface $d\mathcal{S}$. Now we simply need to integrate over the entire surface in order to determine the total flow through the system,

$$\oint_{\mathcal{S}} \rho\vec{v}\cdot d\vec{\mathcal{S}}.$$

Note that this is the integral a vector field (the velocity) dotted with a normal surface vector ($d\vec{\mathcal{S}}$) around a closed surface ($\mathcal{S}$), which means that we can apply the divergence theorem and convert our surface integral into a volume integral,

$$\oint_{\mathcal{S}} \rho\vec{v}\cdot d\vec{\mathcal{S}} = \int_{\mathcal{V}} \left[\nabla\cdot(\rho\vec{v})\right]\, d\mathcal{V}. \tag{9}$$

So now we have an expression involving the total mass flowing out of the system in terms of a volume integral, which is useful because we can also easily express the total decrease in mass of the system as a volume integral. Since the total mass flowing out should equal the total mass lost, we equate these integrals. The volume integral of density yields the mass and taking a time derivative of this integral gives us the rate of change of mass over time,

$$-\frac{\partial}{\partial t}\int_{\mathcal{V}} \rho\, d\mathcal{V} = -\int_{\mathcal{V}} \frac{\partial}{\partial t}\rho\, d\mathcal{V}. \tag{10}$$

Since our integral is only over spatial variables and the density is assumed to be a continuous function, we can exchange the order of the time derivative and the integral. Now we can equate our two integrals and simplify,

$$\begin{aligned}\int_{\mathcal{V}} \nabla\cdot(\rho\vec{v})\, d\mathcal{V} &= -\int_{\mathcal{V}} \frac{\partial}{\partial t}\rho\, d\mathcal{V}\\ \int_{\mathcal{V}} \nabla\cdot(\rho\vec{v})\, d\mathcal{V} + \int_{\mathcal{V}} \frac{\partial}{\partial t}\rho\, d\mathcal{V} &= 0\\ \int_{\mathcal{V}} \left[\nabla\cdot(\rho\vec{v}) + \frac{\partial}{\partial t}\rho\right] d\mathcal{V} &= 0.\end{aligned} \tag{11}$$

The only way for the integral to be zero in general is if the integrand is zero. By setting our integrand to zero and expanding our derivatives we obtain the Eulerian time derivative of the density,

$$\frac{\partial\rho}{\partial t} + \rho(\nabla\cdot\vec{v}) + \vec{v}\cdot\nabla\rho = 0. \tag{12}$$

The Eulerian analysis fixes the coordinate system in space and tracks the motion of the system through that frame. Lagrangian analysis fixes the coordinate system with respect to the object and tracks the system in that frame. In order to formulate the Lagrangian derivative we can use the fact that an infinitesimal change in position $d\vec{r}$ is equal to the velocity multiplied by an infinitesimal change in time $\vec{v}dt$. Starting from the definition of the full derivative, we obtain the Lagrangian derivative of density

$$\begin{aligned}d\rho &= \frac{\partial\rho}{\partial t}dt + d\vec{r}\cdot\nabla\rho\\ d\rho &= \frac{\partial\rho}{\partial t}dt + \vec{v}dt\cdot\nabla\rho\\ \frac{d\rho}{dt} &= \frac{\partial\rho}{\partial t} + \vec{v}\cdot\nabla\rho\end{aligned} \tag{13}$$

If we compare Eqn. 13 with Eqn. 12 we can see that the Lagrangian time derivative of density is given by

$$\frac{d\rho}{dt} = -\rho(\nabla\cdot\vec{v}). \tag{14}$$

So now we have an expression relating the rate of change of density of the gas to the velocity at which the gas is moving.

3. *Fluid Dynamics under Pressure*

Since our hydrogen gas cloud is under pressure, we now extend our analysis of fluid dynamics to describe fluids under pressure. In general, pressure can be described as force per unit area, $P = F/A$. In our situation it will be more useful to consider force to be pressure multiplied by area, $F = PA$. The particular area that we are interested in is the surface area $\mathcal{S}$ of our arbitrary volume. The total force on the surface $\mathcal{S}$ due to pressure is

$$F = -\oint_{\mathcal{S}} P\, d\mathcal{S} = -\int_{\mathcal{V}} \nabla P\, d\mathcal{V}, \tag{15}$$

where we once again have applied the theorem relating closed surface integrals to volume integrals. From this equation, the force per unit volume due to pressure is $-\nabla P$. From here we can apply Newton's second law, $\vec{F} = m\vec{a} = m(d\vec{v}/dt)$. Since we have the force per unit volume we must also divide the right hand side of Newton's second law by unit volume to use our result. Luckily, the mass per unit volume is simply the density so our expression for Newton's second law becomes

$$-\nabla P = \rho\frac{d\vec{v}}{dt} \implies -\frac{\nabla P}{\rho} = \frac{d\vec{v}}{dt}. \tag{16}$$

Next we can apply a process similar to the formulation of the density derivatives in the previous section. For our Eulerian derivative, we have

$$\frac{d\vec{v}}{dt} = \frac{\partial\vec{v}}{\partial t} + (\nabla\cdot\vec{v})\vec{v}. \tag{17}$$

Now we can substitute our expression for the time derivative of velocity developed in Eqn. 16 to see that

$$\frac{\partial\vec{v}}{\partial t} + (\nabla\cdot\vec{v})\vec{v} + \frac{\nabla P}{\rho} = 0, \tag{18}$$

assuming that pressure is the only source of force on our gas cloud. It is also important to remember that our gas cloud is also under a force due to gravity. In order to account for this we must subtract the inward acceleration due to gravity $\vec{g}$ from our previous equation,

$$\frac{\partial \vec{v}}{\partial t} + (\nabla \cdot \vec{v})\vec{v} + \frac{\nabla P}{\rho} - \vec{g} = 0. \tag{19}$$

Now we have an equation relating the acceleration of the particles in the gas cloud. From here we can derive an expression for the minimum radius that a gas cloud must have in order to condense into a gas cloud.

4. Gravitational Collapse

In order to examine the gravitational collapse of the gas cloud we will assume that our cloud is approximately spherical with some radius r and initial uniform density ρ_0. Suppose that some spherical shell within the cloud becomes more dense than the rest of the cloud. We can write down the expression for the radial acceleration of the cloud using Eqn. 19. The first two terms of this equation constitute the shell,

$$0 = \frac{\partial \vec{v}}{\partial t} + (\nabla \cdot \vec{v})\vec{v} + \frac{\nabla P}{\rho_0} - \vec{g}$$

$$0 = \frac{d^2\vec{r}}{dt^2} + \frac{\nabla P}{\rho_0} - \vec{g} \tag{20}$$

$$\frac{d^2\vec{r}}{dt^2} = -\frac{\nabla P}{\rho_0} + \vec{g}. \tag{21}$$

Since all of these vectors are collinear, we can consider only magnitude,

$$\frac{d^2 r}{dt^2} = -\frac{\|\nabla P\|}{\rho} + g. \tag{22}$$

Since our gas cloud is spherical we have azimuthal and polar symmetry. This symmetry mean that the gradient of the pressure should only have a radial dependence. We will approximate the radial dependence of ∇P by P/r. Also, due to the equation of state previously derived we know that P/ρ_0 is a constant that we defined to be a_0^2. By substituting we can see that

$$-\frac{\|\nabla P\|}{\rho} \approx -\frac{P}{r\rho_0} = -\frac{a_0^2}{r}, \tag{23}$$

and thus our acceleration equation becomes

$$\frac{d^2 r}{dt^2} \approx -\frac{a_0^2}{r} + g. \tag{24}$$

Next we will calculate the acceleration due to gravity. Recall our equation for the generalized law of universal gravitation with n dimensions of gravity,

$$F_n = \mathcal{G}_n \frac{m_1 m_2}{r^{n-1}} \frac{\Gamma\left[\frac{n}{2}+1\right]}{n\pi^{n/2}}. \tag{25}$$

In this problem we will let m_1 be the mass of the spherical shell and m_2 be the mass of the rest of the cloud, which is essentially the total mass of the cloud since we are taking a small shell. The mass of the cloud m_2 is $4\pi r^3\rho/3$. In order to obtain the acceleration due to gravity g we simply need to substitute for m_2 and divide by m_1,

$$g = \mathcal{G}_n \frac{4\pi\rho_0}{3r^{n-4}} \frac{\Gamma\left[\frac{n}{2}+1\right]}{n\pi^{n/2}}. \tag{26}$$

So the acceleration of our spherical shell is

$$\frac{d^2 r}{dt^2} \approx -\frac{a_0^2}{r} + \mathcal{G}_n \frac{4\pi\rho_0}{3r^{n-4}} \frac{\Gamma\left[\frac{n}{2}+1\right]}{n\pi^{n/2}}. \tag{27}$$

In order for the shell to collapse into a star we need the radial acceleration to be negative,

$$\frac{d^2 r}{dt^2} < 0$$

$$-\frac{a_0^2}{r} + \mathcal{G}_n \frac{4\pi\rho_0}{3r^{n-4}} \frac{\Gamma\left[\frac{n}{2}+1\right]}{n\pi^{n/2}} < 0$$

$$\frac{a_0^2}{r} < \mathcal{G}_n \frac{4\pi\rho_0}{3r^{n-4}} \frac{\Gamma\left[\frac{n}{2}+1\right]}{n\pi^{n/2}}$$

$$\frac{3a_0^2}{4\mathcal{G}_n\pi\rho_0} \frac{n\pi^{n/2}}{\Gamma\left[\frac{n}{2}+1\right]} < r^{5-n}$$

$$\left(\frac{3a_0^2}{4\mathcal{G}_n\pi\rho_0} \frac{n\pi^{n/2}}{\Gamma\left[\frac{n}{2}+1\right]}\right)^{1/(5-n)} < r. \tag{28}$$

We define the **Jean's Radius** r_j to be the minimum radius required for a spherical gas cloud to collapse into a star,

$$r_j = \left(\frac{3a_0^2}{4\mathcal{G}_n\pi\rho_0} \frac{n\pi^{n/2}}{\Gamma\left[\frac{n}{2}+1\right]}\right)^{1/(5-n)}. \tag{29}$$

Another more commonly referenced quantity is the **Jean's Mass** of a gas cloud, which is derived from the Jean's Radius using the relationship between volume mass and density,

$$M_j = \frac{4}{3}\pi\rho_0 \left(\frac{3a_0^2}{4\mathcal{G}_n\pi\rho_0} \frac{n\pi^{n/2}}{\Gamma\left[\frac{n}{2}+1\right]}\right)^{3/(5-n)}. \tag{30}$$

We can then substitute in the constants which equal a_0^2 to obtain our final expression for the Jean's Mass,

$$M_j = \frac{4}{3}\pi\rho_0 \left(\frac{3}{4\mathcal{G}_n\pi\rho_0} \frac{kT}{\mu m_H} \frac{n\pi^{n/2}}{\Gamma\left[\frac{n}{2}+1\right]}\right)^{3/(5-n)}. \tag{31}$$

We now have a single expression that places a lower bound on the mass required for a gas cloud in order to collapse into a star. Note that for a given choice of dimension, the only non-constants in the equation for the Jean's mass are the initial density and temperature of the gas cloud.

TABLE I: Parameter Settings for each Calculation Trial

Trial	ρ	T
1	ρ_{min}	T_{min}
2	ρ_{min}	T_{max}
3	ρ_{max}	T_{min}
4	ρ_{max}	T_{max}

III. DATA AND ANALYSIS

The Jean's mass has two input parameters: temperature and density. So given an initial temperature and density, the minimum mass required for a hydrogen gas cloud to condense into a star can be calculated. Realistic initial temperatures are between $T_{min} = 10$ and $T_{max} = 30$ K and realistic starting densities are between $\rho_{min} = 10^{10}$ and $\rho_{max} = 10^{12}$ hydrogen masses per unit volume [1]. Given this variation in the ranges of realistic values for our input parameters, the Jean's mass was calculated at four different settings of temperature and density as summarized in Table I.

The Jean's mass was calculated for values of n between one and nine at each of the four settings of input parameters from Table I, excluding $n = 5$ due to the discontinuity in Eqn. 31. The Jean's masses are plotted versus the$n$ in Fig. 3. Notice how the graph exhibits a radical behavior with a vertical asymptote at 5, which is to be expected given the discontinuity. It is also worth mentioning that the trials that are on the extremes of the ranges for any given choice of n are the two trials with a mix of maximum and minimum initial parameters.

The goal of this investigation was to determine if a modified force gravitational force law would be able to account for star formation as viewed in three dimensions. The force law has been modified to allow for different radial exponents. As shown in Fig. 3, the minimum mass required for a cloud to form a star varies greatly

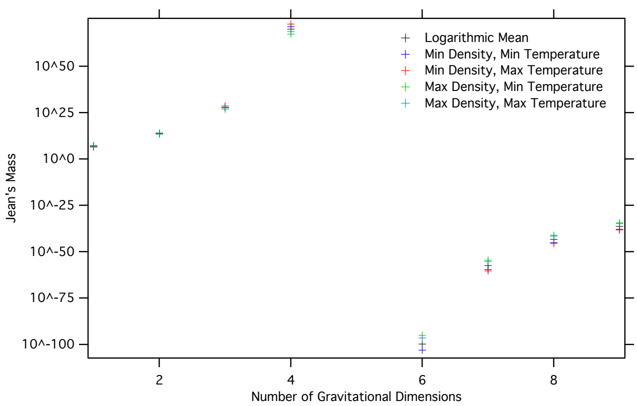

FIG. 3: Pplot of Jean's Mass versus n. Since the Jean's mass varies by over 100 orders of magnitude, a logarithmic scale is used for the abscissa.

TABLE II: Order of Magnitude of $\mathcal{G}$ for choices of n

n	$\log_{10}[\mathcal{G}]$
1	-38
2	-23
4	4
6	33
7	48
8	62
9	76

depending on radial dependence. For dimensions other than three, I want to recover the behavior we see when $n = 3$. In order to do so I will change the gravitational constant in the other dimensions. I set the Jean's mass equation for $n \neq 3$ equal to the value of the Jean's mass for $n = 3$ and solved for what the gravitational constant would have to be in order for the Jean's Mass when $n \neq 3$ to equal the Jean's Mass when $n = 3$. The order of magnitude for the new values of $\mathcal{G}$ are shown in Table II. These new values of $\mathcal{G}$ differ by 114 orders of magnitude as the dimension varies from $n = 1$ to $n = 9$.

IV. CONCLUSION

Our original question was simple: are the laws of physics as we know them unique? The answer to this question is incredibly complex and we as a community of physicists are nowhere near approaching the full answer. But we can approach pieces of this problem. When considering the problem of forming a star, we could choose a completely different radial dependence than the $1/r^2$ dependence that we see in everyday life. Using these new radial dependences, we were able to calculate the Jean's mass for given initial conditions. We were able to then calculate the order of magnitude that we would have to change the gravitational constant to in order to recover the behavior that we see in three dimensions.

However, if we were to use these altered force laws in other systems, such as dropping a ball on the Earth's surface, they would break down. If we were to calculate something as simple as the acceleration due to gravity at the Earth's surface with these modified force laws, these calculations would be inaccurate. Developing alternate theories to explain the phenomena that we know can be explained by the current accepted physical theories is a challenging yet exciting problem.

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