Chaos in a Driven, Nonlinear Electrical Oscillator: Determining Feigenbaum's Delta

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M.J. Feigenbaum has shown that all systems that exhibit a period doubling route to chaos share a common scaling function, such that $\delta = (V_2 - V_1)/(V_3 - V_2) = 4.669...$ This value, δ, is called "Feigenbaum's delta." A current driven by a function generator, traveling through a resistor, inductor, and a diode, acts as a driven nonlinear electrical oscillator, and is shown to exhibit a period doubling route to chaos. This system was used to experimentally determine δ, producing an average value of δ =5.0±1.0, consistent with the theoretical value.

INTRODUCTION

Humans exploring their world at first tried to find the order in all physical systems; however, there were some systems which seemed completely unpredictable. These systems were labeled "random". Upon closer observation, it was found that many ordered systems show extreme sensitivity to their initial conditions, while seemingly random systems exhibit a sort of order over time. This state somewhere in between periodic and completely unpredictable was termed "chaos".[1]

In 1976 Mitchell Feigenbaum found a theory that connected the many systems that exhibit a period doubling route from the ordered to chaotic states with a continuous change in parameter values [1]. Included in this category are mechanical and chemical oscillators, the electrical oscillator that will be discussed in this paper, and certain population models. These systems all share the characteristic of recursiveness, in which successive iterations of a function rely on previous states of the function[1]. He found a relationship in which the details of a recursive equation become irrelevant, and out falls a constant called "Feigenbaum's delta." Feigenbaum's delta measures the ratio of the distance between successive period doublings, and has been found theoretically and experimentally to be a constant value[1-4].

It is one of the systems that exhibits period doubling that is studied in this experiment. A driven, nonlinear, electric oscillator shows bifurcation at critical values of current frequency or amplitude[4]. From this, Feigenbaum's delta is found, and can be compared to theoretical values.

THEORY

A system is considered simple if it is predictable, that is, able to be modeled exactly by a mathematical equation. Between systems that are simple and those that are completely random are those called "chaotic." Feigenbaum's theory of period doubling can be used to describe the behavior of the electric oscillator subjected to a periodic voltage. At some range of parameter values, for example, the amplitude of the applied signal, the system repeats itself every period of time T. After reaching a critical value for voltage, the system begins to repeat itself every interval 2T. It is at this value that it is said that the period has doubled. This occurs repeatedly, requiring 4T, 8T, and 16T to reproduce itself as voltage increases. The value denoting the periodicity will be defined as the number of units of time T it takes for the system to repeat itself. It is also necessary to define transition values for the bifurcation of the system. These will be defined with the equation[2]

$$V_n \equiv 2^n T \qquad (1).$$

A transition value V_n will denote the period *to which a system is moving.* For example, as the system moves from period two to period four, the transition value will be labeled V_2. Eventually, it is found that the period has doubled ad infinitum, denoted $V\infty$, and at that point the behavior is no longer periodic, and is considered chaotic.[2]

The universal behavior among all systems that exhibit period doubling is as follows:

$$\delta_n \equiv \frac{V_{n+1} - V_n}{V_{n+2} - V_{n+1}} \qquad (2),$$

δ_n converges to a constant value δ = 4.6692016...[2].

EXPERIMENT

The experimental setup and procedure are very similar to an experiment done previously by Paul S. Linsay[4]. The current produced by a Hewlett Packard function generator flows through an inductor in which L = 0.1805±0.0001 mH, a resistor in which R = 4.513 ± 0.003 Ω, and a diode, which provides the nonlinear element of the circuit. The function generator is connected to channel 1(x) of a Hewlett Packard oscilloscope (model 54600B), and the voltage across the diode, V_0, is connected to channel 2(y). The circuit diagram is shown below, in Figure 1.

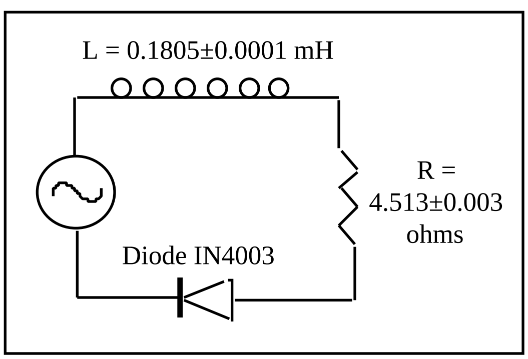

Figure. 1. Circuit diagram of setup for a driven nonlinear electrical oscillator. A function generator creates a current that travels through a conductor, resistor, and a diode.

A sine function is applied to the system by the function generator, which also controls the variables--amplitude and frequency--that change the behavior of the diode. There are two methods by which period doubling can be observed--either by holding the amplitude constant and increasing the frequency, or by holding the frequency constant and increasing the amplitude. First, an amplitude/frequency combination was found which clearly demonstrated that the system was period one. Next, it was determined which variable would remain constant and which would change. The changing variable was be increased by fairly large increments until it bifurcated, showing period two. Figure 2 below shows a phase diagram of the circuit in period two. The variable was then decreased, and adjusted by smaller increments to more precisely determine the point at which period doubling occurred. It should be noted that it was impossible to determine the precise value at which period doubling occurred, and so it was necessary to determine a range of values which could be considered the critical range. The value midway between the high and low values was considered the critical point, with one half of the difference between high and low values considered the error in the critical value.

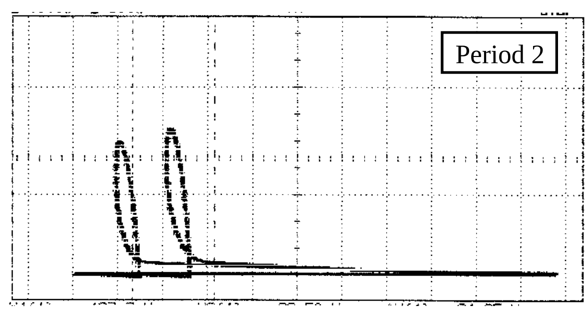

Figure 2. Phase space diagram of V_0 versus V_S at Frequency = 180.0 KHz and Amplitude = 2.985 V_{pp}. Period 2 is seen here as two peaks.

Period doubling was observed in this manner from period one to two, two to four, fourto eight, and sometimes eight to sixteen.

The critical values determined from these observations were used to find the value for δ. Also observed were periods of chaos, as well as periodic regions within the chaos that had odd periods and multiples of odd periods where 3T, 6T, and 5T were needed for the system to reproduce itself. These periods did not double enough to determine a value for δ inside the chaotic regime, however.

An important consideration in determining critical values of period doubling is that, over time, peaks were observed to separate. At values very near the critical value, the system would spontaneously bifurcate from period one to two over time. It was suspected that this was a thermal effect. This was proven when the system, which had just bifurcated, was cooled with ice enclosed in plastic, and observed to return to period one. The separation was observed over a period of six minutes, and the peaks in a period two state were observed to separate at an average rate of 0.057±0.002 V/min. Because the effect is apparently linear, a trial conducted at any time after the apparatus is turned on should be accurate; however, a trial should be conducted as quickly as possible to reduce the error caused by thermal effects.

ANALYSIS AND INTERPRETATION

Eight trials were conducted, collecting the critical values for period 1 to period 8, and to period 16 where possible. Four of the trials were completed with a constant amplitude, and four with a constant frequency. Sample data appears for trial 5 below, in Table 1.

Period change	Frequency (KHz)
V_1 (1 - 2)	348.03±0.13
V_2 (2 - 4)	678.24±0.26
V_3 (4 - 8)	752.86±0.33
V_4 (8 - 16)	766.95±0.69

Table I. Critical values for Trial 5, in which Amplitude remained constant, A = 4.000 V_{pp}.

with the values from Table I, Feigenbaum's delta is calculated as shown below, using equation 2.

$$\delta_{1-3} = \frac{V_2 - V_1}{V_3 - V_2} = \frac{678.24 - 348.03}{752.86 - 678.24} = 4.43$$

and $\delta_{2\text{-}4}$ = 5.30, where $\delta_{1\text{-}3}$ denotes that critical values for the calculation were taken for periods one, two, and four, and in $\delta_{2\text{-}4}$ critical values were taken for periods two, four, and eight. The same calculation was carried out for the variable amplitude trials. All of the data are presented below, in Tables II and III.

Trial Number	$\delta_{1\text{-}8}$
1	4.19±0.20
6	4.00±0.07
7	4.19±0.28
8	4.16±0.29

Table II. Values of Feigenbaum's delta calculated for trials in which frequency remained constant. In all trials frequency = 180.0 KHz, except for trial 7, in which frequency = 200.0 KHz.

In addition, δ for period doubling from 2-16 was calculated for trials 4 and 5 to be 4.91±0.44 and 5.30±0.39, respectively.

It can be seen in Table III that error values for $\delta_{2\text{-}16}$ are much higher than those for $\delta_{1\text{-}8}$. This can be attributed to the fact that, with successive bifurcations, it becomes much more difficult to distinguish the point at which the period has doubled. Three of the four error values in Table II are much higher than expected; however, reasons for this are unknown.

Trial Number	$\delta_{1\text{-}8}$
2	7.46±0.01
3	5.71±0.02
4	5.66±0.03
5	4.43±0.04

Table III. Values for Feigenbaum's delta for trials in which amplitude remained constant. In trial 2, A = 4.618 V_{pp}, in trial 3 and 4, A = 4.000 V_{pp}, and in trial 5, A = 3.000 V_{pp}.

As seen in Table II and III, the values for δ are in agreement with one another, and fairly close to the theoretical value δ =4.669. Table III shows that trial two did not yield consistent value for δ, with the experimental value δ =7.46±0.01, 60% higher than the theoretical value. The other trials yielded more consistent results, with the value of trial 4, $\delta_{2\text{-}16}$ in excellent agreement with the theoretical value. Averaging all trials together yields a value of δ =5.0±1.0, which is consistent with the published value.

CONCLUSION

Eight trials were run, in order to find several values of Feigenbaum's delta. All results were on the same order of the theoretical value δ =4.669. Most were reasonably close to this value, with one 60% higher than it should be, and one falling within experimental error of the theoretical value. The average value for all eight trials yields δ =5.0±1.0, which is consistent with the published theoretical value. In the future, precautions could be taken to regulate the temperature of the circuit in order to obtain more accurate results.

[1]James Gleick, Chaos: Making a New Science, (Viking, New York, 1988), p. 171-181, 306-307.

[2]Mitchell J. Feigenbaum, "Universal Behavior in Nonlinear Systems," Physica D 7, 16-39, (1983).

[3]Mitchell J. Feigenbaum, "Quantitative Universality for a Class of Nonlinear Transformations," J. Stat. Phys 19 (1), 25-52, (1978).

[4]Paul S. Linsay, "Period Doubling and Chaotic Behavior in a Driven Anharmonic Oscillator," Phys. Rev. Letters 47 (19), 1349-1352, (1981).