Measuring Feigenbaum's δ in a Bifurcating Electric Circuit

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A sinusoidal voltage at a controlled amplitude and frequency was applied to a nonlinear, diode circuit. The peaks of the voltage output, measured across the nonlinear diode, underwent period doubling when varying either the frequency or amplitude of the input signal. By comparing the frequency values at sequential bifurcations, Feigenbaum's δ was calculated to be 4.28±0.03, after 3 doublings. This value varies by 8.3% from the accepted [1] universal value for δ of 4.669..., but corresponds, within error, to the value of 4.3±0.1 determined in previous, published [2] diode circuit experiments.

Introduction:

Many physical systems are inherently complex, seemingly random, yet deterministic. Humans have typically tried to control and simplify these systems to suit their needs, that is, to tame nature. Processes exist in many systems where the varying of a single parameter in a presumably simple, linear system can produce behavior indicative of complex, nonlinear systems, yet is still governed by the same deterministic rules. An example of this is varying the parameter called temperature. A pot of still water can be forced to exhibit periodic and nonperiodic convection currents and other phenomena consistent with boiling as the temperature is increased to from 5 - 100°C. One particular approach to chaos is period-doubling.

In the case of period-doubling, a sinusoidal signal of period 1 is applied to a system, the output of that system initially resembles that of the input, but as the value of a single parameter is changed, the period of the output will bifurcate, or double (1, 2, 4, 8, 16 ...). This pattern will continue to increase on to infinity yielding an infinite period or chaos (Fig 1). With each bifurcation, the amount the parameter needs to be changed to produce a bifurcation decreases in a constant way. One can measure the sequential changes in the parameter and compare one to the next. This ratio is the constant called Feigenbaum's δ. This process is exhibited not only in theory and numerical simulations, but also in many experimental systems [1] including hydrodynamic, electronic, laser, and acoustic systems.

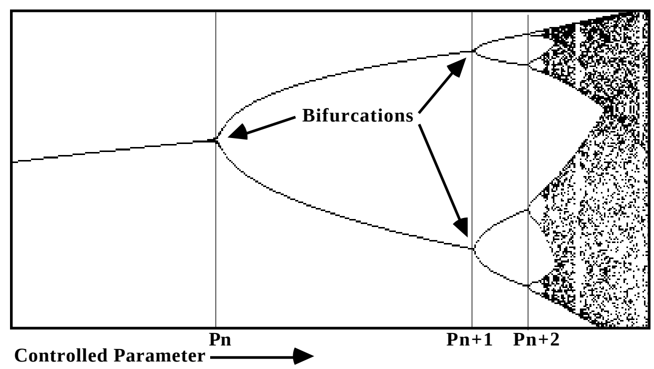

Figure 1 : An example of a numerically calculated bifurcation diagram exhibiting period doubling as the variable on the horizontal axis is varied.

Theory:

The chaotic circuit (Fig 2), a variation of Hilborn's [3] consists simply of a sinusoidal input voltage signal, an inductor, and a diode providing the nonlinearity for the system. These pieces give the system the degree of freedom necessary to produce chaos. The output signal is measured as the voltage drop across the diode. This signal evidences period-doubling bifurcation and chaos by varying either the frequency or amplitude of the input signal. Exactly why this occurs may not be readily apparent. Hilborn [3] describes a model where water is traveling through plumbing in a manner similar to electrons through the wires of the circuit. In this model, the incoming signal forces or attempts to slosh the water back and forth at a specific frequency and amplitude. The resistor simply resists this flow in either direction. The inductor acts as a sort of propeller. As the water flows back and forth, the propeller builds up speed and momentum resisting the flow of water as it is pushed backwards by the input signal until it reverses and again builds up speed in the opposite direction. The diode can be pictured as a flap valve in the pipe, which allows water to pass it in one direction, but quickly closes as the water begins to reciprocate. The output signal can be viewed as the periodic motion of the flap valve opening and closing. As the frequency of the input signal increases, the flap valve can no longer completely close before the next wave of water strikes, forcing it open again. In this manner, the output signal can attain higher periodicity than the input signal. The results are similar for increasing amplitude with a constant frequency.

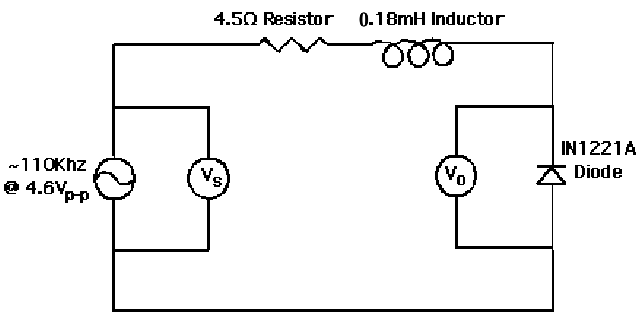

Figure 2 : Diagram of the chaotic circuit where V_S is the applied input voltage and V_O is the output voltage.

To calculate Feigenbaum's δ in a chaotic circuit, one must use the equation [2]

$$\delta \equiv \frac{P_{n+1} - P_n}{P_{n+2} - P_{n+1}}$$

where P_n is the value of the varied parameter at a bifurcation, P_{n+1} and P_{n+2} are the values of that parameter at the next two consecutive bifurcations, and δ is the constant rate of change of the parameter between bifurcations (Fig 1). In the case of the electronic circuit, the values of P_n correspond either to signal frequency or amplitude values depending on which parameter is varied to produce period-doubling while the other is held constant.

Setup:

The nonlinear circuit is assembled on a breadboard as indicated in Figure 2. A SRS-DS335 function generator is connected to the circuit in order to generate the sinusoidal input signal. The 100 MHz HP 54600B oscilloscope monitors the applied voltage, V_S. and the voltage V_O across the diode.

Procedure:

The first thing to consider while running this experiment is that the output signal drifts with time and the function generator does not always yield the signal it is displaying on its front panel, making rapid, consistent measurements, made from the oscilloscope, very important for an accurate result. The drift may be caused by heating in the circuitry, thus varying the resistivity of the nonlinear diode. By directly measuring the frequency of the signal from the function generator, its stability was guaranteed as the amplitude was changed. The amplitude of the signal on the oscilloscope did not correspond to that displayed by the function generator as its frequency was changed. For consistency, the amplitude had to be adjusted on the function generator with any change in frequency . Using the reading from the oscilloscope yielded the most accurate results with an error of $0.03V_{p-p}$. Two methods were employed by which period-doubling in the circuit could be observed: ramping the frequency or the amplitude, holding the other constant in each case. Recording the P_n parameter value on the oscilloscope at increasing bifurcations produces the data needed to calculate Feigenbaum's δ.

The information on the oscilloscope can be displayed in two ways useful for data aquisition. The first is the time series which shows how the output signal changes with time (Fig 3). With a period 1 signal, all the peaks are at the same height. As the modified parameter is increased, the heights of the peaks begin to diverge forming patterns 2, 4, 8 ... times the length of the period 1 input signal. The second is the phase space plot where the applied signal is plotted against the output in an x-y manner (Fig 4). During period doubling, new loops form in the image. The number of loops matches the periodicity of the output signal. It is most useful to watch for bifurcations in the phase space plot and take measurements in time series mode.

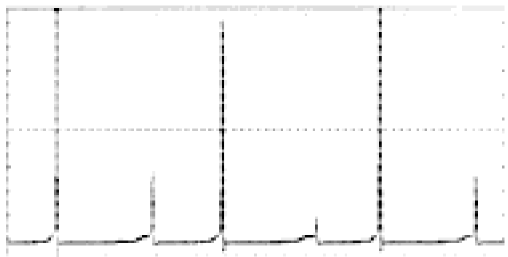

Figure 3 : An output signal time series with four distinct peaks showing a period four times that of the applied signal.

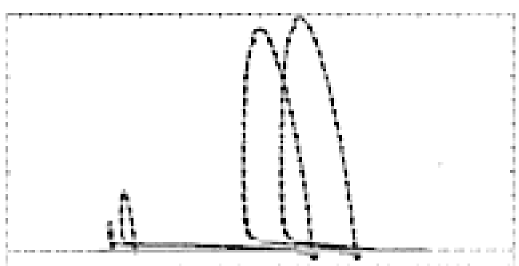

Figure 4 : The same signal as Figure 3 displayed in phase space, V_0 vs V_S. The number of loops corresponds to the periodicity of the output signal in relation to the input.

Analysis and Interpretation:

Period-doubling bifurcations were observed between the frequencies of 110-135KHz at an amplitude of $6V_{p\text{-}p}$ as measured by the oscilloscope (Table 1). By varying the signal frequency in this range, periods of 2, 4, and 8 were distinguishable before noise in the circuit began to dominate the output signal.

Frequency (KHz)	**Period**
114.54±0.01	1 - 2 transition
127.69±0.01	2 - 4 transition
130.76±0.01	4 - 8 transition

Table 1 : The value for frequency at the period 2, 4, and 8 bifurcations while holding the amplitude constant at $6V_{p\text{-}p}$.

Substituting into the Feigenbaum equation,

$$\delta = \frac{(127.69 - 114.54)}{(130.76 - 127.69)} = \frac{13.15}{3.07} = 4.28$$

with an error of 0.65% or 4.28±0.03 for Feigenbaum's δ.

When repeating the procedure for changing the amplitude parameter, the frequency was held at a constant 130KHz (Table 2). At this value, period-doubling bifurcation was observed between voltage amplitudes of 5-7 $V_{p\text{-}p}$ as measured on the oscilloscope. Varying the amplitude again yielded only periods of 2, 4, and 8.

Amplitude ($V_{p\text{-}p}$)	**Period**
5.19±.03	1 - 2 transition
6.53±.03	2 - 4 transition
6.81±.03	4 - 8 transition

Table 2 : The value for the amplitude at period 2, 4, and 8 bifurcations while holding the frequency constant at 130KHz.

Again substituting into Feigenbaum's equation,

$$\delta = \frac{(6.53 - 5.19)}{(6.81 - 6.53)} = \frac{1.34}{.28} = 4.79$$

with an error of 21% giving an answer of 4.8±1.0 for Feigenbaum's δ.

Data for a qualitative diagram was recorded for a comparison between this circuit's behavior with those of previous experiments (Fig 5). Here, the frequency was held constant at 130KHz. From left to right, the output signal shows period one oscillation until the first bifurcation at 5.98 $V_{p\text{-}p}$ where the signal cycles with period 2 between two peaks which now vary in height. This continues with periods of 4 and 8 before the noise of the system causes the periodicity to become indeterminate. Beyond this point, windows of periodicity appear through the noise of the circuit and extremely large periodicities including 7, 6, 4, and finally 2 in that order. Due to the length of time required to take the data, this data is not equivalent to the data in Table 2, but can be qualitatively compared with Hilborn's diode bifurcation diagram which undergoes a similar period-doubling route to chaos (Fig 6).

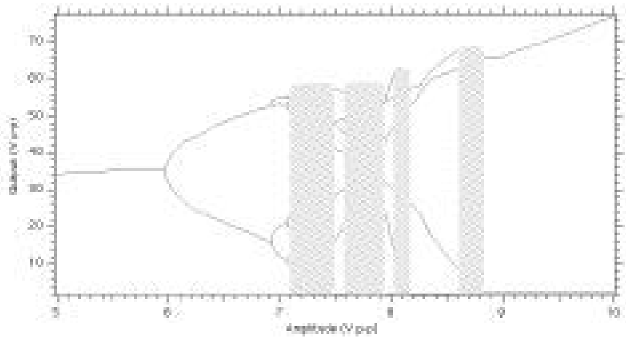

Figure 5 : A qualitative bifurcation diagram for the chaotic circuit. The vertical scale displays the relative heights of peaks measured from the time series plot as the amplitude is increased on the horizontal scale.

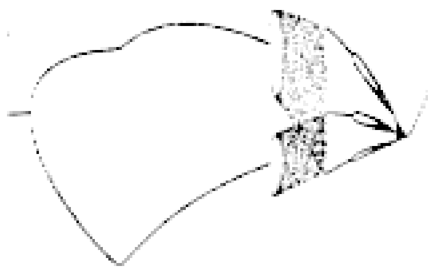

Figure 6 : Hilborn's bifurcation diagram for a chaotic circuit.

Conclusion:

Both values obtained from this experiment, 4.28±0.03 and 4.8±1.0, are consistent, within error, both with each other and with other experimental data for nonlinear diode circuits. The value for δ from amplitude variation was not nearly so precise as that from frequency variation due to the relatively large error compared to the measured value whereas the accuracy in the frequency was far greater. Also, the qualitative diagram for the circuit's bifurcation diagram showed a strong comparison with that from previous data indicating true period-doubling and nonlinear behavior. This experiment could be improved by attempting to reduce noise in the circuit and increasing the accuracy and consistency of the function generator.

[1] Mitchell J Feigenbaum, "Universal Behavior in Nonlinear Systems" Los Alamos Science v1 p4-27, 1980.

[2] H-O.Peitgen, H. Jürgens, D.Saupe, Fractals for the Classroom (Springer-Verlag New York Inc., New York,1992).

[3] Robert C. Hilborn, Chaos & Nonlinear Dynamics : An Introduction for Scientists & Engineers, (Oxford University Press, Oxford, 1994).