

Stefan-Boltzmann Law

Austin R. Carter

Department of Physics, The College of Wooster

Wooster, Ohio 44691, USA

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The tungsten filament of a light bulb was used in a thermal experiment as an approximate blackbody to verify the Stefan-Boltzmann law. The resistance of the filament at room temperature was measured to be $0.309 \pm 0.002 \Omega$. This was then used in conjunction with tabulated CRC resistivity values to calculate the temperature of the filament for different power inputs. Two analysis techniques were employed to confirm that the radiation from an object is proportional to the fourth power of its absolute temperature. The first method used a three parameter, fourth-order polynomial function and determined the proportionality constants $\sigma \epsilon A_s$ and $k A_o$ to be $(5.80 \pm 0.09) \times 10^{-13} (\frac{W}{K^4})$ and $(1.186 \pm 0.068) \times 10^{-3} (\frac{W}{K})$, respectively. The second method used a log-log plot of the inputted power as a function of the absolute temperature and determined $\sigma \epsilon A_s$ to be $(6.32 \pm 0.09) \times 10^{-13} (\frac{W}{K^4})$ which is in agreement with the first method's value to 3σ . The exponent was determined to be 4.00 ± 0.55 which strongly supports the Stefan-Boltzmann law.

I. INTRODUCTION & THEORY

In 1879 Josef Stefan experimentally observed that the power per unit area of a blackbody is proportional to the fourth power of its absolute temperature. This same relationship was theoretically derived from Maxwell's theory and classical thermodynamics in 1884 by Ludwig Boltzmann and is therefore called the Stefan-Boltzmann law.

However, classical thermodynamics broke down when Lord Rayleigh and Sir James Jeans attempted to use electromagnetism to describe the energy density distribution for a blackbody. Although their theory worked relatively well for low frequencies, it failed when applied to high frequencies; the intensity diverged to infinity! This unfeasible result was dubbed the *ultraviolet catastrophe* because ultraviolet light was the highest frequency radiation known at the time.[1] In 1900, Max Planck boldly proposed a solution to the problem. By making the *ad hoc* assumption that atoms behave like electrical oscillators, absorbing and emitting only discrete amounts of energy for a range of frequencies, Planck was able to accurately describe the energy density distribution observed for blackbody radiation. Furthermore, the Stefan-Boltzmann law can be independently derived from Planck's equations.[2]

Planck remedied the ultraviolet catastrophe with an energy distribution function now called *Planck's law*:

$$u(\lambda) = \frac{8\pi hc\lambda^{-5}}{e^{\frac{hc}{\lambda kT}} - 1}. \quad (1)$$

Where $u(\lambda)$ is the energy density ($\frac{J}{m^3}$), h is Planck's constant ($6.626 \times 10^{-34} J \cdot s$), c is the speed of light ($299,792,458 \frac{m}{s}$), λ is the wavelength of the radiation (m), k is the Boltzmann constant ($1.381 \times$

$10^{-23} \frac{J}{K}$), and T is the absolute temperature (K).

The total energy density (U) is obtained by integrating Eq. 1 over all wavelengths.

$$U = \int_0^\infty \frac{8\pi hc\lambda^{-5}}{e^{\frac{hc}{\lambda kT}} - 1} d\lambda$$
$$U = \frac{8\pi^5 k^4}{15h^3 c^3} T^4. \quad (2)$$

This is the result that Stefan originally observed in 1879; for an ideal blackbody, the radiation per unit area is proportional to fourth power of the absolute temperature:

$$U = \sigma T^4 \quad (3)$$

Where σ is a proportionality constant known as *Stefan's constant* equal to $5.6705 \times 10^{-8} (\frac{W}{m^2 \cdot K^4})$. Note that U is solely dependent on the temperature.[2]

However, the radiation per unit area for non-ideal objects is decreased by factors such as surface composition and color. These contribute to a new *emissivity* factor, ϵ :

$$U = \frac{P}{A_s} = \epsilon \sigma T^4$$
$$P = \epsilon \sigma A_s T^4 \quad (4)$$

Where P is the power, A_s is the surface area of the object (m^2) and ϵ is the unit-less emissivity with a value ≤ 1 . Thus, Eq. 4 is the power emitted by an approximate blackbody.

Consider the filament of a lightbulb: an approximate blackbody. For an object in thermal equilibrium, the inputted power must be equal to the outputted power. A filament has two sources of power input and output. Power into the filament is provided from the electrical current running through the resistor (P_e) and the radiation absorbed from

ambient room temperature (P_o). Power out of the filament includes electromagnetic radiation (P_r) and any power lost to conduction (P_c).

$$\begin{aligned} P_{in} &= P_{out} \\ P_e + P_o &= P_r + P_c \end{aligned} \quad (5)$$

P_e can be expressed in terms of the current of the circuit and the potential difference across the filament. Eq. 4 relates P_o and P_r to the absolute ambient temperature (T_o) and the absolute equilibrium temperature (T_f), respectively. The power lost to conduction (P_c) can also be related to these two temperatures using:

$$P_c = kA_o(T_f - T_o). \quad (6)$$

Where k is a thermal conductivity constant dependent on the material ($\frac{W}{m \cdot K}$) and A_o is a constant dependent on the physical contact with other materials (m).[3] Substituting these equations into Eq. 5 and rearranging to solve for P_e gives

$$P_e = IV = \sigma \epsilon A_s T_f^4 - \sigma \epsilon A_s T_o^4 + kA_o(T_f - T_o) \quad (7)$$

Where the T_f^4 term is the blackbody radiation, the T_o^4 term is the blackbody absorption, and the $(T_f - T_o)$ term is the power lost to conduction.

II. EXPERIMENT

This experiment utilized a *Pasco* Stefan-Boltzman Lamp (TD-8555), a *Keithley* auto-ranging voltmeter (197), a *Keithley* auto-ranging ammeter (197), a *Hewlett Packard* DC Power Supply (6290A), and a *Hewlett Packard* Power Supply (6214A). The lamp is a light bulb with a tungsten filament and has a threshold of up to 13 volts or 3 amps. The power supply (6290A) is capable of providing 0-40 volts and up to 3 amps. The power supply (6214A) has considerably more precision and is capable of providing 0-12 volts and up to 1.2 amps. Fig. 1 is a schematic diagram of the experimental setup.

The first part of the experiment determined the resistance of the tungsten filament at room temperature by sending small currents ($\approx 0.1 - 3.0$ mA) through the lamp using the 6214A power supply. After the readings on the voltmeter and ammeter settled (≈ 10 s) their values were recorded. In the second part of the experiment the 6214A power supply was removed and the 6290A power supply was used to send relatively large currents ($\approx 0.5 - 3.0$ A) through the filament. The lamp illuminated at ≈ 1.1 amps. The readings on the voltmeter and ammeter were allowed to quickly settle (≈ 5 s) before their values were recorded. The lamp was allowed to cool for about 30 seconds between measurements to help reduce systematic error caused by conduction.

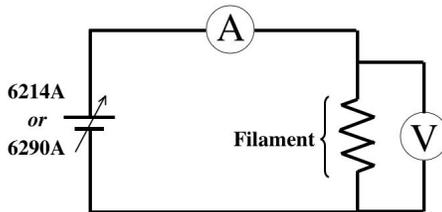


FIG. 1: A circuit schematic of the experimental setup. The resistor (filament) is in a *Pasco* Stefan-Boltzman lamp, the ammeter is connected in series, and the voltmeter is connected in parallel to the lamp electrodes to ensure an accurate measurement. The two power supplies, 6214A and 6290A, were interchanged into the experimental setup; only one was used at a time.[4]

III. RESULTS AND ANALYSIS

A. Measuring Filament Temperature

The resistance (R) of an object is related to the resistivity (ρ) of the object's material according to

$$R = \rho \frac{L}{A_c}, \quad (8)$$

where L is the object's length (m) and A_c is the object's cross-sectional area (m^2).[3] It is important to note that resistance is object-specific while resistivity is a temperature dependent physical property of a substance. The dependence of resistivity on absolute temperature for tungsten has been well tabulated.[5] Consequently, if the resistivity of the filament is known, its temperature can be determined. That is to say that the the resistivity can act as a type of thermometer for the filament.

Dividing Eq. 8 at $T_f > T_o$ by Eq. 8 at $T_f = T_o$ produces

$$\frac{R_f}{R_o} = \frac{\rho_f}{\rho_o}. \quad (9)$$

This is based on the assumption that A_c and L are constant, but these terms increases as the temperature increases due to thermal expansion. The change in length due to expansion compared to the total length is equal to

$$\frac{\Delta l}{l} = \alpha \Delta T. \quad (10)$$

where α is the thermal rate of expansion for tungsten equal to $5 \times 10^{-6}(K^{-1})$, and l is any dimensional

length of the filament. For the maximum change in temperature for this experiment, $\Delta T \approx 3000K$, $\frac{\Delta l}{l} \approx 15 \times 10^{-3} = 1.5\%$. This effect is negligible and therefore the assumption that L and A_c are constant is justified.

Manipulation of Eq. 9 produces

$$\rho_f = \frac{\rho_o V_f}{R_o I_f}, \quad (11)$$

where ρ_o and ρ_f are the reference resistivity at room temperature and the final resistivity at some final temperature ($m\Omega \cdot cm$), respectively, R_o is the reference resistance of the filament at room temperature (Ω), and V_f and I_f are the voltage and current at the final temperature in V and A respectively.

At room temperature ($300K$), ρ_o for tungsten is $5.65 m\Omega \cdot cm$, but R_o is unknown and must be determined. This was done by sending a very small current through the filament. Such a small current produces very little power such that $T_f \approx T_o$. By plotting these voltages as a function of current (Fig. 2), a weighted best fit line was produced with a slope equal to the reference resistance (R_o) at room temperature.

With the value of R_o firmly established, the resistivity of the filament can be calculated using Eq. 11 by varying the current and voltage. This resistivity value can then be used to determine the temperature of the filament using the tabulated values in the CRC. [5]

A best fit to the tabulated values was created (See Fig. 3) using a quadratic function of the form:

$$y = C_2 x^2 + C_1 x + C_o. \quad (12)$$

The fit yielded the equation

$$T_f = -0.0498\rho_f^2 + 35.84\rho_f + 129.1 \quad (13)$$

where ρ_f is the final resistivity ($m\Omega \cdot cm$) and T_f is the final temperature of the filament (K).

B. Testing Stefan-Boltzmann Law

By plotting the power inputted into the filament as a function of the absolute temperature, the Stefan-Boltzmann law can be tested using Eq. 7 (See Fig. 4). A fourth order polynomial of the form

$$y = C_4 x^4 + C_3 x^3 + C_2 x^2 + C_1 x + C_o \quad (14)$$

was fitted to the data. The x^3 and x^2 terms were forced to zero to put the function in the form of Eq. 7.

This fit yielded the equation

$$P_e = (5.80 \times 10^{-13})T_f^4 + (1.1861 \times 10^{-3})T_f - 0.41811 \quad (15)$$

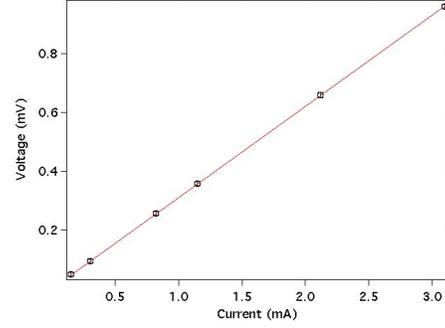


FIG. 2: Small currents were used to probe the filament at room temperature. The slope of the weighted best fit line is equal to the resistance of the filament at room temperature and has a value of $0.309 \pm 0.002\Omega$. This value did not change appreciably ($< 0.3\%$) when the intercept was forced to zero.

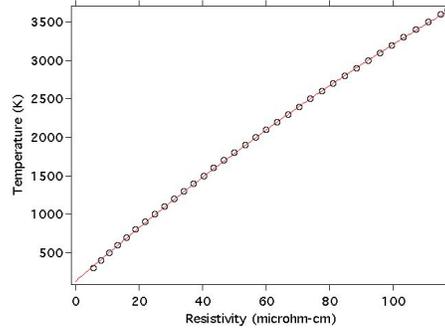


FIG. 3: The tabulated dependence of resistivity on absolute temperature was fitted to a quadratic function. The values of the coefficients were recorded with appropriate uncertainty and used to determine the temperature of the filament.

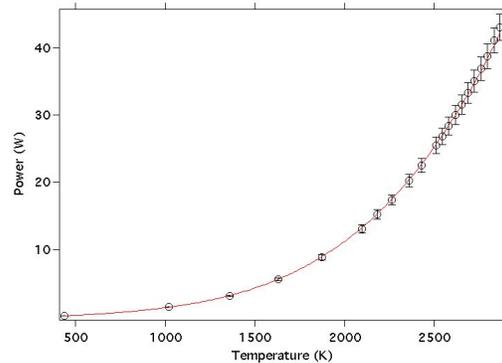


FIG. 4: As expected, the inputted power as a function of absolute temperature data fits well ($\chi^2 = 1.41$ for 21 points) to a three parameter, fourth order polynomial. The systematic increase in uncertainty for the larger values of the temperature can be attributed to a systematic increase in the random error of the temperature.

The values of the coefficients of Eq. 15 were used to determine the value of the constants $\sigma\epsilon A_s = (5.80 \pm 0.09) \times 10^{-13} \left(\frac{W}{K^4}\right)$ and $kA_o = (1.186 \pm 0.068) \times 10^{-3} \left(\frac{W}{K}\right)$.

Looking at Eq. 15, it becomes apparent that for large values of the absolute temperature ($T_f > 2200K$) the radiation term (T_f^4) dominates. The other two terms contribute $< 17\%$. Therefore Eq. 7 becomes

$$P_e \approx \sigma\epsilon A_s T_f^4. \quad (16)$$

Taking the natural log of this approximation

$$\ln(P_e) \approx \ln(\sigma\epsilon A_s) + 4\ln(T_f) \quad (17)$$

Thus if the Stefan-Boltzmann law holds, then $\ln(P_e)$ should be a linear function of $\ln(T_f)$ with a slope of four. (See Fig. 5) The slope yielded a value for the exponent of 4.00 ± 0.55 while the intercept yielded a value of $\sigma\epsilon A_s = (6.32 \pm 0.09) \times 10^{-13} \left(\frac{W}{K^4}\right)$.

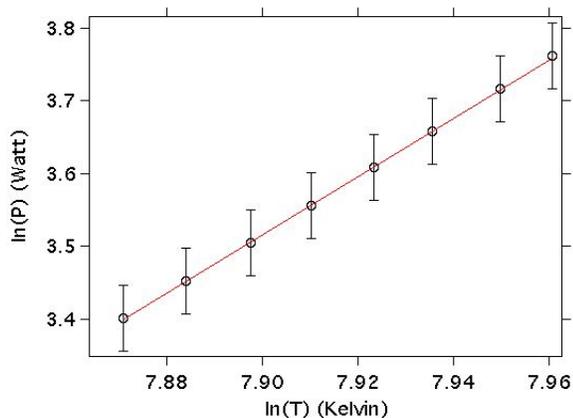


FIG. 5: As expected, the natural log of the input power as a function of the natural log of the absolute temperature fits well ($\chi^2 = 0.0084$ for 8 points) to a linear function. The slope is equal to the exponent of the power and has a value of 4.00 ± 0.55 . This relatively large uncertainty can be attributed to the large error in the temperature but the close proximity of the best fit to the data gives supports to a higher level of certainty.

The first method used to confirm the Stefan-Boltzmann law employed a three parameter, fourth-order polynomial function to describe the behavior of the filament. The function fit well to the data with a χ^2 value of 1.41 for 21 points. Since there was a good fit to the data, the coefficients had relatively low uncertainty. This allowed the proportionality constant $\sigma\epsilon A_s$ to be determined to be $(5.80 \pm 0.09) \times 10^{-13} \left(\frac{W}{K^4}\right)$ with an uncertainty of $< 2\%$ and the constant kA_o to be determined to be $(1.186 \pm 0.068) \times 10^{-3} \left(\frac{W}{K}\right)$ with an uncertainty of $< 6\%$. The second method used a log-log plot of the input power as a function of the absolute temperature and determined $\sigma\epsilon A_s$ to be $(6.32 \pm 0.09) \times 10^{-13} \left(\frac{W}{K^4}\right)$ which has an uncertainty of $< 2\%$ and is in agreement with the first method's value to 3σ . The power exponent was also determined to be 4.00 ± 0.55 which has an uncertainty $< 14\%$.

The observed systematic increase in uncertainty for the larger values of the temperature can be attributed to a systematic increase in the random error for large values of temperature. Additionally, for the first method, the data appears to systematically increase for large T values above the best fit curve. An explanation of this could be thermal expansion. Thermal expansion would increase the surface area as the filament's temperature increased. This in turn would cause the curve to become more steep. The effect, however, appears slight. In addition, the temperature of the glass bulb is not the same as the room temperature. The bulb's temperature increases as it is turned on and off. This in turn will make the absorption term in Eq. 7 a larger factor. Nonetheless, the increased room temperature would have a minimal effect on Eq. 7 for large values of T_f and can therefore be considered negligible.

The quality of the fits for both methods, the agreement of the exponent in the second method, and the correspondence of the two values of the constant $\sigma\epsilon A_s$, creates a strong argument for the support of Stefan-Boltzmann's law.

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