Bicycle Stability,
Is the Steering Angle Proportional to the Lean?

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Abstract

If the steering of a bicycle is proportional to the lean angle, then the motion of the center of mass of the bike can be modeled as a damped simple harmonic oscillator. This would in part explain why a bicycle is stable. An experiment was performed to determine weather or not the steering angle is proportional to the lean. Due to noisy data, the proportionality was not conclusively verified, but evidence does *suggest* that the steer angle is proportional to the lean angle for small angles. The constant of proportionality was determined to be $k = 2.40 \pm 0.15$. Improvements for future versions of this investigation are suggested.

Introduction

Most people can ride a bicycle, but few, if any, can explain why. Examining the question leads to the identification of a number of forces, torques, linear accelerations and angular accelerations, which can be used to develop a mathematical expression useful in attempting to gain some insight into the problem.

Among those who have considered the problem (1-5), it seems to be widely accepted that when executing a turn, a rider must balance gravitational forces with the pseudo-centrifugal force generated by turning the handlebars and traveling in an arc. When a rider attempts to turn to the right, for example, he must lean into the turn, that is, to the right. To avoid falling over onto his right side, he must steer to the right so that the bike travels a curve of such a radius as to generate a centrifugal force that 1.) exactly balances the gravitational force and continues the turn without falling over, or 2.) generates a centrifugal force of great enough magnitude to un-lean the bike, returning it to its upright balanced position. Whether the bike continues in the turn or is up-righted depends on the magnitude of the steer angle implemented by the rider. Anyone who has spent some time on a bicycle will probably agree that this argument seems reasonable. Additionally, a bicycle whose front wheel is locked in place, unable to steer, is virtually un-rideable[4].

The above argument fails to explain why a bicycle will remain upright for as much as 20 seconds or so if it is allowed to roll freely. It also fails to explain why a bike feels extremely stable when it is moving at very high speeds. Attempts to explain this stability through torque arguments (associated with the spin of the front wheel) by references 1-3 and 5 were refuted by D.E.H. Jones[4] who actually built what he called an "unrideable-bicycle" by attaching a second wheel to the front of a bike whose purpose it was to cancel the torque created by the spinning of the first wheel. The problem with the unrideable bicycle was that it was quite rideable[4].

Another possible reason for the stability of the bicycle follows from the equation of motion of the center of mass of a bike rider system (eq 5). If the lean angle of a bicycle is proportional to the steering angle, the equation of motion for the center of mass of the bike-rider pair reduces to the equations of motion for a damped simple harmonic oscillator. This could lead to yet another explanation for the stability of the bicycle.

Theory

J. Lowell uses the following method to develop a mathematical expression for the dynamics of a bicycle in motion. The center of mass of the bike-rider pair has acceleration given by the expression

$$a_{total} = h\ddot{\theta} + b\ddot{\eta} + v^2/R \qquad (1)$$

Where h, θ, η, b, v and R are illustrated in Fig. 1.
From Fig 1,

$$R\dot{\eta} = v \qquad (2)$$

And the component of weight normal to the frame is

$$w_{\perp} = mg\sin\theta \approx mg\theta \qquad (3)$$

Where g is the acceleration due to gravity. Dividing both sides of (3) by m, equating the result to (1) and then making the substitution expressed by (2) gives

$$g\theta = h\ddot{\theta} + b\frac{d}{dt}\left({}^{v}\!/_{R}\right) + \left({}^{v^2}\!/_{R}\right) \qquad (4)$$

Putting $R\alpha = a$ into (4),

$$\ddot{\theta} + \left({}^{bv}\!/_{ha}\right)\dot{\alpha} + \left({}^{v^2}\!/_{ha}\right)\alpha - \left({}^{g}\!/_{h}\right)\theta = 0 \qquad (5)$$

If the steering angle α is assumed to be proportional to the lean angle θ, then (5) reduces to the equation for a damped simple harmonic oscillator with the form

$$\ddot{\theta} + \left({}^{bv}\!/_{ha}\right)k\dot{\theta} + \left(\left({}^{v^2}\!/_{ha}\right) - \left({}^{g}\!/_{h}\right)\right)k\theta = 0 \qquad (5a)$$

Which is consistent with the form of the differential equation for a damped SHO.

If experimental evidence supports this assumption (α=kθ), then the stability of the bicycle can be partially explained by the bike behaving as a damped simple harmonic oscillator[1].

To verify or reject the assumption, experimental evidence is needed. A simple plot of α vs. θ for a bike coasting through a turn should provide the needed data.

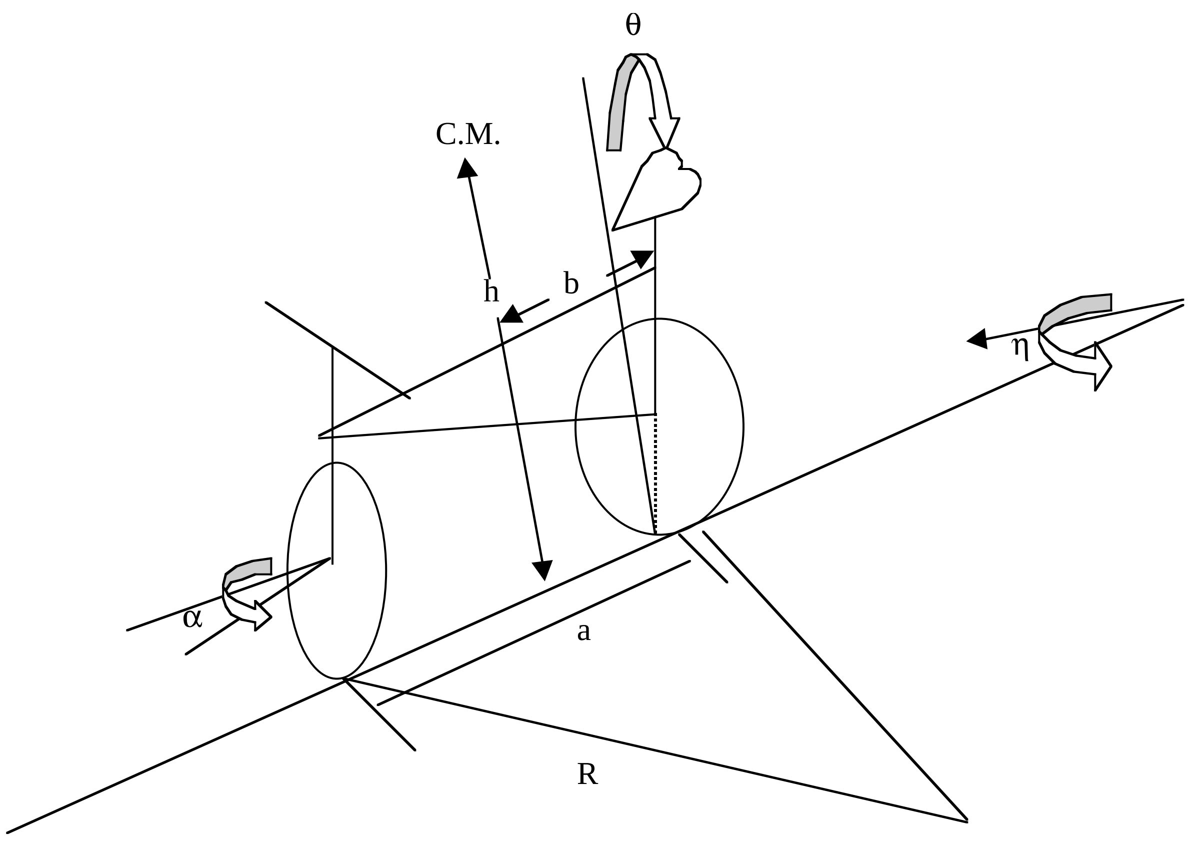

Fig 1. (From source 1) Bicycle and pertinent variables and parameters. η is the angle that a line connecting the contact points of the two wheels makes with some arbitrary horizontal reference vector.

Setup

To record the lean and steering angle data needed for this investigation, a Sakar TR-2L Camera mount was affixed to a 2.5 cm diameter wooden dial rod and replaced the bicycle seat. The dial rod was slightly too narrow for the quick release mechanism of the bicycle to grip it firmly, so wooden wedges were added to increase the effective width of the rod.

The camera was mounted to the bicycle via a modified camera mount and aligned such that the full width of the brake levers (used as reference marks) could be seen for any angle that the bars were turned.

The room in which data were taken was large enough to comfortably ride a bicycle at relatively low speeds (two to three meters per second). Yarn was hung from the ceiling to a few centimeters above the floor. At the end of each piece of yarn, a 50 g mass was attached. These hanging strands were used as lines of reference to determine the lean angle of the bicycle.

Procedure

A number of reference lines were created by hanging red strands of yarn from the ceiling. The camera was set to record and the bike was ridden towards the first hanging strand of yarn and then turned to the right, passing several more strands before finally being straightened out. Essentially, the bike was ridden through a 180° turn past the array of hanging strands. For the data that were analyzed, the bike was always within two meters of a strand of yarn.

The movie of the run was loaded into iMovie 2.1.1. Only clips in which the top tube of the bike was aligned with a nearby vertical were used. The video analyses program, Videopoint 2.1 was used to record the coordinates of four points per frame for the 31 frames of the run that appeared to be well in-line with a vertical in the room.

Data/Analysis

The steer angle was determined by capturing images of the handlebars at known angles and plotting the projected distance between the brake levers in pixels (determined by analysis with videopoint 2.1) vs. the cosine of the corresponding angle. The equation of the line fit to the data was used to determine the steer angle of the bicycle during the run by solving for θ in the equation (from fig 2),

$$d = a + b\cos\alpha$$

$$\alpha = \cos^{-1}\left(\frac{d-a}{b}\right) \qquad (7)$$

Where d is the projected length of the handlebars on the screen in pixels (the *projection* warranted the use of Cosα).

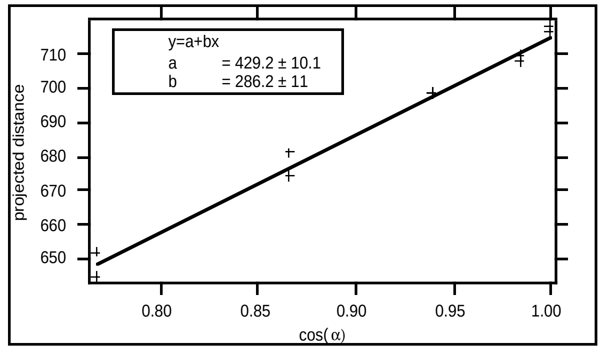

Fig 2. Projected distance between brake levers vs. cos(α). Distance is in pixels.

The lean angles of each frame were determined by assuming that the camera was oriented vertically on the bicycle and remained so throughout the course of the run. A vertical line in the reference frame of the room translated to a diagonal line in the frame of the camera when the camera was leaning (i.e. the bike was leaning). A simple analysis of the geometry of the setup shows that the lean angle of a reference line equals the lean angle of the bike. The x and y coordinates of two points on each reference line were recorded for each frame to determine the lean angle.

Initially yarn was not implemented and vertical lines in the room (corners, window frames etc.) were used for reference. This method yielded few points and they were largely spaced in the path of the bike. Additionally, it appears that there is a significant amount of uncertainty associated with using far away (more than three or four meters) lines of reference. Limiting the data to the above restrictions, the first runs were disregarded because all of the first runs used many far away reference lines. The first analysis of the data used data points from multiple runs, rather than one steering sequence. It was desirable to use data from one continuous run for plots of steering and lean angles vs. time because they should result in trends that can be associated with the leaning and steering sequence associated with riding a bike through a single turn.

To produce a set of acceptable data, the bicycle was ridden standing up (to avoid sitting on the camera) in an arc that passed near the hanging strands of yarn. 31 frames were chosen from this run because in these frames the top tube of the bicycle was well aligned with a nearby line of reference. Because the requirements for useable data were not fully realized until very late in the research process, only data for one run were analyzed after the yarn was hung (although several movies were recorded). The chosen run was used because it represented the smoothest bicycle arc with smooth transitions during the changing of the steering angle.

The run for which data were analyzed (previously and henceforth referred to as "the run") resulted in a large amount of noisy data (Fig 3). By examining the clips corresponding to data that did not appear to behave as expected, it was determined that far away reference lines resulted in lean angle values that were not accurate. It was also verified that nearby reference lines (Fig 4) resulted in points that behaved more consistently.

Due to resolution limitations in the video that were probably the result of insufficient lighting, some of the yarn was not visible and far away reference lines were used for several of the points in the middle of the sequence of the 31 selected frames. Specifically, frames 4 through 6 (Frames 1 through 3 used far away reference lines as the bike approached the yarn) and frames 28 through 31 were used to create the data that were of crucial importance to this investigation (Plotted in Fig 4). The discarded frames all relied on far away reference lines.

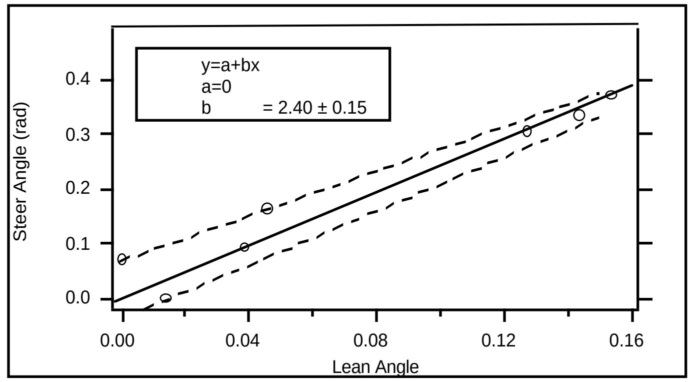

Fig 4. Steer angle versus lean angle. The plot displays a relationship between Steer angle and lean angle. Dotted l represent bounds of region in which the actual line should

Results/Discussion

Much of the data in Fig. 3 appears to be noisy. The beginning and end of each run appear to behave as expected. The bike begins its path with a very small lean angle and a very small steer angle. As the turn progresses, the steer and lean angles increase. The steer angle is always larger in magnitude than the lean angle. This seems to fit with what one would expect from experience with bike riding.

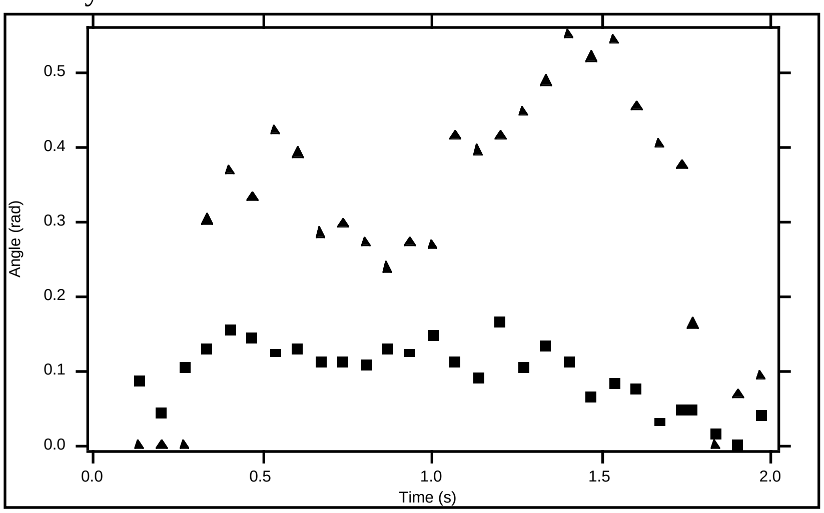

Fig 3. Steering angle (triangular markers) and Lean angle (square) with respect to time. The time units are in seconds, but are arbitrary

A similar plot to Fig. 3 is provided in an article by J. Fajans. Fig. 3 agrees with Fajans's plots for small and large times. Trends in the middle of the plot are in agreement, but the data are much noisier in Fig. 3 as a result of some phenomena causing the relatively large uncertainty in the lean angles determined by far away reference lines.

The cause of the uncertainty associated with the large reference lines in unknown. It is likely that it is a result of the keystoning effect associated with wide angle optical lenses.

Uncertainties in the values of the steering angle are largely a result of poor planning. The markings used to determine the changing projected length of the handlebars as the steer angle changed were the points at which the break lever mounts connected with the top of the handlebars. While using Viewpoint to collect data for the final run, it was noticed that this point was obscured as the bars were rotated. The location of the point on the right end of the bar appeared to be located slightly to the left (along the bar) of where it actually was. This probably skewed the projected length data but the effect was definitely small (No more than 6 to 8 pixels for the largest steering angles).

The purpose of the experiment was to determine whether or not the steer angle was proportional to the lean angle. After discarding the data corresponding to far away reference lines, a plot of steer angle vs. lean angle was made (Fig 4). Although probably not sufficient to conclude that steer angle is necessarily proportional to lean angle, the plot is quite suggestive that this is true for small angles. The only inferences that can be made from Fig. 4 must be made for small angles as the steering and lean angles at the beginning and end of the runs (the only data plotted in Fig 4) were the smallest angles of the run. The constant of proportionality *k*, was determined to be 2.40±0.15. The uncertainty of this value was determined by Igor Pro. The dashed lines in Fig 4 represent where one would expect the dashed lines to lie (based off of the most deviant end data points). Calculation of the slopes of the dashed lines agrees fairly well with the uncertainty in slope quoted by Igor.

Conclusions

Fig. 4 suggests that the steering angle of a bicycle is proportional to the lean angle, however, due to the limited number of useable data points in this experiment, it cannot be positively concluded.

A modified version of this project could produce more conclusive results. Improvements would include the use of spotlights to allow for better visibility of the yarn reference lines, and a faster frame rate. This would increase the number of available data points per run as well as decrease the uncertainty associated with using distant reference lines. The certainty in the steer angle values could be increased by painting 2 small marks onto the handlebars that are visible throughout the entire turning sequence.

According to J. Lowell, if k, in equation (5a) is greater than ga/v^2, stability is ensured. For this investigation, no attempt was made to record instantaneous velocity and thereby calculate the value of the quantity ga/v^2 for the data points. With velocity measurements, it would be possible to measure the horizontal length a between the seat and the center of mass of the system and compare ga/v^2 *to* k. This would be difficult because the center of mass probably changes position as the rider shifts his/her weight.

ACKNOWLEDGMENTS

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