The Gravitational Potential Between Two Line Segments

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The gravitational potential energy for two line segments is derived and simplified in terms of geometric vectors. In the limit that the length of one of the line segments goes to zero the previously know potential energy is verified. As the length of the second line segment goes to zero Newton's formula for the potential energy between two point masses is derived. The kinetic energy for the system is derived and the Lagrangian formulation is used to show that 3 out of the 6 coordinates needed to describe the system are ignorable.

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INTRODUCTION

Central to the study of celestial dynamics is the three body problem [1, 2]. If only two point mass bodies are involved, the problem can be solved exactly, but the addition of a third body makes the equations unsolvable and numerical techniques are required. An understanding of the intricacies requires greater perseverance.

When the two body problem is extended it is often unsolvable and may display interesting phenomena such as chaos. For example, if the two bodies are confined in a spherical universe, arbitrarily complex orbits can occur [3]. By inverting Newton's third law and preserving Newton's second law, the dynamics of the two body problem can be chaotic [4]. By extending one of the point masses into a line segment both order and chaos can be demonstrated. This is known as the slash-dot problem [5].

The study of asteroids is one phenomenon that requires research in two body problems. Unlike moons, planets, and stars, when studying the dynamics of asteroids, which are very small and irregularly shaped, it is inadequate to assume point masses. As energy can be stored in rotational degrees of freedom of these irregularly shaped asteroids, angular momentum exchange can unbind orbits causing asteroid escape [6]. Escape in this manner is not possible within the classical two body problem. To be able to predict the motion of these asteroids, it is necessary to understand the gravitational dynamics with the point mass assumption removed.

The study of the gravitational dynamics between two line segments is examined in [7, 8]. The universe is restricted to only two dimensions yet the dynamics seem chaotic.

THE POTENTIAL ENERGY

Derivation

Newton proposed that the gravitational potential energies between two point masses is

$$V = -\frac{Gm_Am_B}{r}, \tag{1}$$

where G is the gravitational constant, m_A and m_B are the masses of the particles, and r is the distance between the particles. The gravitational energy between two line segments is then

$$V = -\frac{Gm_Am_B}{L_AL_B}\int_{-L_B/2}^{L_B/2}\int_{-L_A/2}^{L_A/2}\frac{1}{\|\boldsymbol{\delta_A}-\boldsymbol{\delta_B}\|}\,\mathrm{d}\lambda_\mathrm{A}\,\mathrm{d}\lambda_\mathrm{B}, \tag{2}$$

where L_A and L_B are the lengths of the slashes, $\boldsymbol{\delta_A}$ is a vector pointing from the origin to an arbitrary point on slash A and $\boldsymbol{\delta_B}$ is a vector pointing from the origin to an arbitrary point on slash B. In terms of λ_A and λ_B,

$$\boldsymbol{\delta_A} = \mathbf{r_A} + \lambda_\mathrm{A}\hat{\mathbf{R}}_\mathbf{A}, \tag{3a}$$

$$\boldsymbol{\delta_B} = \mathbf{r_B} + \lambda_\mathrm{B}\hat{\mathbf{R}}_\mathbf{B}, \tag{3b}$$

where $\{\mathbf{r_A}, \mathbf{r_B}\}$ point from the origin to the center of mass of each slash and $\{\hat{\mathbf{R}}_\mathbf{A}, \hat{\mathbf{R}}_\mathbf{B}\}$ are unit vectors pointing in the direction of each slash. These vectors are shown in Fig. 1. By varying λ_A and λ_B over the full length of each slash all possible values of $\|\boldsymbol{\delta_A}-\boldsymbol{\delta_B}\|$ may be realized.

Use polar coordinates to derive values for $\|\boldsymbol{\delta_A}-\boldsymbol{\delta_B}\|$ in terms of $\{r, \phi, \varphi_A, \varphi_B\}$. These coordinates are shown in Fig. 2, where r is defined to be the distance between the center of masses of the slashes,

$$\mathbf{r} = \mathbf{r_A} - \mathbf{r_B}, \tag{4a}$$

$$r = \|\mathbf{r}\|, \tag{4b}$$

ϕ is the polar angle with respect to the $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$ axis,

$$\phi = \arctan\left(\frac{r_y}{r_x}\right), \tag{5}$$

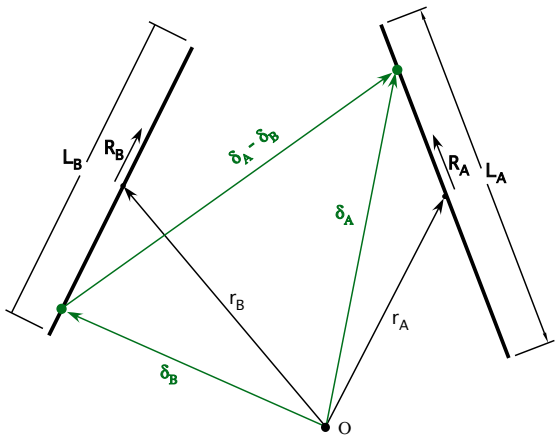

FIG. 1: Several useful vectors to define the position of each slash.

and $\{\varphi_A,\ \varphi_B\}$ are the angles of the slashes with respect to the $\{\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}\}$ axis,

$$\varphi_A = \arctan\left(\frac{R_{A\phi}}{R_{Ar}}\right), \tag{6a}$$

$$\varphi_B = \arctan\left(\frac{R_{B\phi}}{R_{Br}}\right). \tag{6b}$$

Unlike the $\{\mathbf{x}, \mathbf{y}\}$ basis vectors the $\{\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}\}$ basis vectors may change with time.

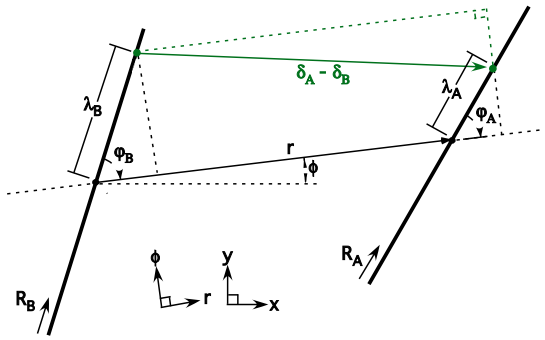

FIG. 2: The coordinates are used to express $\|\boldsymbol{\delta_A} - \boldsymbol{\delta_B}\|$.

By examining Fig. 2 one can determine that

$$\delta_{Ar} - \delta_{Br} = r + \lambda_A \cos\varphi_A - \lambda_B \cos\varphi_B, \tag{7a}$$

$$\delta_{A\phi} - \delta_{B\phi} = \lambda_A \sin\varphi_A - \lambda_B \sin\varphi_B. \tag{7b}$$

What is convenient about the choice of coordinates is that the potential energy is independent of ϕ. When the problem is examined in the Lagrangian formulation ϕ becomes ignorable.

With some manipulation we can write Eq. 2 in the form

$$V = -K \int_{-L_B/2}^{L_B/2} \int_{-L_A/2}^{L_A/2} \frac{1}{\sqrt{\alpha + \beta\lambda_A + \lambda_A^2}}\, d\lambda_A\, d\lambda_B, \tag{8a}$$

$$K = \frac{G m_A m_B}{L_A L_B}, \tag{8b}$$

$$\alpha = r^2 + \lambda_B^2 - 2r\lambda_B \cos\varphi_B, \tag{8c}$$

$$\beta = 2r\cos\varphi_A - 2\lambda_B \cos(\varphi_A - \varphi_B). \tag{8d}$$

Use the identity

$$\int \frac{dx}{\sqrt{\alpha + \beta x + x^2}} = \log\left(\beta + 2x + 2\sqrt{\alpha + \beta x + x^2}\right), \tag{9}$$

to integrate with respect to λ_A. With some more algebra the integral may be written as

$$V = -K \int_{-L_B/2}^{L_B/2} (U - L)\, d\lambda_B, \tag{10a}$$

$$U = \log\left(-m_U \cos\Delta + d_U \sin\Delta + \sqrt{m_U^2 + d_U^2}\right), \tag{10b}$$

$$L = \log\left(-m_L \cos\Delta + d_L \sin\Delta + \sqrt{m_L^2 + d_L^2}\right), \tag{10c}$$

where

$$\Delta = \varphi_A - \varphi_B, \tag{11a}$$

$$m_U = \lambda_B - \frac{L_A}{2}\cos\Delta - r\cos\varphi_B, \tag{11b}$$

$$m_L = \lambda_B + \frac{L_A}{2}\cos\Delta - r\cos\varphi_B, \tag{11c}$$

$$d_U = \frac{L_A}{2}\sin\Delta - r\sin\varphi_B, \tag{11d}$$

$$d_L = -\frac{L_A}{2}\sin\Delta - r\sin\varphi_B. \tag{11e}$$

To solve the second integral use the identity

$$\int \log\left(-m\cos\theta + d\sin\theta + \sqrt{m^2 + d^2}\right) dm, \tag{12a}$$

$$= -m + d\csc\theta \log(A_1) + (m + d\cot\theta)\log(A_2), \tag{12b}$$

$$A_1 = m + \sqrt{m^2 + d^2}. \tag{12c}$$

$$A_2 = -m\cos\theta + d\sin\theta + \sqrt{d^2 + m^2}. \tag{12d}$$

The complete analytic form of the potential is long and unenlightening. The full form of the potential can be found in [7].

Geometric Vector Simplification

The complete analytic expression for the potential energy is long and messy. However it is simplified to a

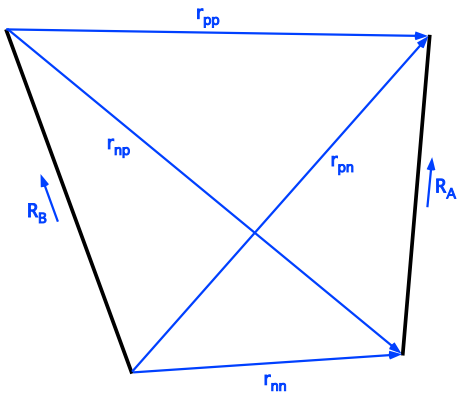

FIG. 3: Useful geometric vectors. The potential energy can be completely specified using these six vectors, the gravitational constant, and the mass density of each slash.

manageable form when geometric vectors are analyzed. Six geometric vectors are shown in Fig. 3. $\mathbf{r_{pp}}$ is a vector that points from the top of slash B to the top of slash A. $\mathbf{r_{pn}}$ is a vector that points from the bottom of slash B to the top of slash A. $\mathbf{r_{np}}$ is a vector that points from the top of slash B to the bottom of slash A. $\mathbf{r_{nn}}$ is a vector that points from the bottom of slash B to the bottom of slash A. The potential energy written in terms of these vectors is

$$V = -K(L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8), \tag{13a}$$

$$L_1 = \|\hat{\mathbf{R}}_\mathbf{B} \times \mathbf{r_{nn}}\| \csc\Delta \log\left[\frac{-\mathbf{r_{nn}}\cdot\hat{\mathbf{R}}_\mathbf{B} + \|\mathbf{r_{nn}}\|}{-\mathbf{r_{np}}.\hat{\mathbf{R}}_\mathbf{B} + \|\mathbf{r_{np}}\|}\right], \tag{13b}$$

$$L_2 = \|\hat{\mathbf{R}}_\mathbf{B} \times \mathbf{r_{pp}}\| \csc\Delta \log\left[\frac{-\mathbf{r_{pp}}\cdot\hat{\mathbf{R}}_\mathbf{B} + \|\mathbf{r_{pp}}\|}{-\mathbf{r_{pn}}.\hat{\mathbf{R}}_\mathbf{B} + \|\mathbf{r_{pn}}\|}\right], \tag{13c}$$

$$L_3 = \|\hat{\mathbf{R}}_\mathbf{B} \times \mathbf{r_{nn}}\| \cot\Delta \log\left[\frac{\mathbf{r_{nn}}\cdot\hat{\mathbf{R}}_\mathbf{A} + \|\mathbf{r_{nn}}\|}{\mathbf{r_{np}}.\hat{\mathbf{R}}_\mathbf{A} + \|\mathbf{r_{np}}\|}\right], \tag{13d}$$

$$L_4 = \|\hat{\mathbf{R}}_\mathbf{B} \times \mathbf{r_{pp}}\| \cot\Delta \log\left[\frac{\mathbf{r_{pp}}\cdot\hat{\mathbf{R}}_\mathbf{A} + \|\mathbf{r_{pp}}\|}{\mathbf{r_{pn}}.\hat{\mathbf{R}}_\mathbf{A} + \|\mathbf{r_{pn}}\|}\right], \tag{13e}$$

$$L_5 = \hat{\mathbf{R}}_\mathbf{B}\cdot\mathbf{r_{pn}} \log\left[\mathbf{r_{pn}}\cdot\hat{\mathbf{R}}_\mathbf{A} + \|\mathbf{r_{pn}}\|\right], \tag{13f}$$

$$L_6 = \hat{\mathbf{R}}_\mathbf{B}\cdot\mathbf{r_{np}} \log\left[\mathbf{r_{np}}\cdot\hat{\mathbf{R}}_\mathbf{A} + \|\mathbf{r_{np}}\|\right], \tag{13g}$$

$$L_7 = -\hat{\mathbf{R}}_\mathbf{B}\cdot\mathbf{r_{pp}} \log\left[\mathbf{r_{pp}}\cdot\hat{\mathbf{R}}_\mathbf{A} + \|\mathbf{r_{pp}}\|\right], \tag{13h}$$

$$L_8 = -\hat{\mathbf{R}}_\mathbf{B}\cdot\mathbf{r_{nn}} \log\left[\mathbf{r_{nn}}\cdot\hat{\mathbf{R}}_\mathbf{A} + \|\mathbf{r_{nn}}\|\right]. \tag{13i}$$

where

$$\csc\Delta = \frac{1}{\|\hat{\mathbf{R}}_\mathbf{B} \times \hat{\mathbf{R}}_\mathbf{A}\|}, \tag{14a}$$

$$\cot\Delta = \frac{\hat{\mathbf{R}}_\mathbf{B}\cdot\hat{\mathbf{R}}_\mathbf{A}}{\|\hat{\mathbf{R}}_\mathbf{B} \times \hat{\mathbf{R}}_\mathbf{A}\|}. \tag{14b}$$

Limits of the Potential

As the lengths of one of the slashes goes to zero the potential energy of the slash-slash body problem should equal the potential of the slash-dot body problem. Because the potential energy is proportional to the mass density of each slash $\{m_A/L_A, m_B/L_B\}$, when one of the slashes goes to a point, the mass density goes to infinity. The value of $L_1+L_2+L_3+L_4+L_5+L_6+L_7+L_8 = L_N$ goes to zero as one of the slashes goes to zero. Therefore the potential energy goes to $0/0$ as one of the slashes goes to zero. L'Hôpital's rule solves the dilemma. Without loss of generality let L_A go to zero, then

$$\lim_{L_A\to 0} V = -\frac{Gm_Bm_A}{L_B}\frac{dL_N}{dL_A}. \tag{15}$$

Simplification yields the solution

$$V = -\frac{Gm_Bm_A}{L_B}\log\left[\frac{-\hat{\mathbf{R}}_\mathbf{B}\cdot\mathbf{r_{pp}} + \|\mathbf{r_{pp}}\|}{-\hat{\mathbf{R}}_\mathbf{B}\cdot\mathbf{r_{nn}} + \|\mathbf{r_{nn}}\|}\right], \tag{16}$$

which is the potential energy derived in [5]. Repeating the same procedure to take the limit as L_B goes to zero yields Eq. 1 which is the potential energy between two point masses.

THE KINETIC ENERGY

The total kinetic energy for each slash is the motion of the center of mass plus the rotation about the center of mass,

$$T = T_A + T_B, \tag{17a}$$

$$T_A = \frac{1}{2}m_A\dot{\mathbf{r}_\mathbf{A}}^2 + \frac{1}{2}I_A\omega_\mathrm{A}^2, \tag{17b}$$

$$T_B = \frac{1}{2}m_B\dot{\mathbf{r}_\mathbf{B}}^2 + \frac{1}{2}I_B\omega_\mathrm{B}^2, \tag{17c}$$

where $\{\omega_A, \omega_B\}$ is the angle of each slash measured from the $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$ axis

$$\omega_A = \phi + \varphi_A, \tag{18a}$$

$$\omega_B = \phi + \varphi_B, \tag{18b}$$

and $\{I_A, I_B\}$ are the moments of inertia for each slash,

$$I_A = \frac{1}{12}m_AL_A^2, \tag{19a}$$

$$I_B = \frac{1}{12}m_BL_B^2, \tag{19b}$$

Make the change of variables to switch to the center of mass frame,

$$T = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 + \frac{1}{2}(I_A\dot{\omega}_{\mathrm{A}}^2 + I_B\dot{\omega}_{\mathrm{B}}^2), \tag{20a}$$

where $\mathbf{R}$ gives the location of the center of mass of the system, μ is the reduced mass, M is the total mass, and

$$\dot{\mathbf{r}}^2 = \dot{r}^2 + r^2\dot{\phi}^2. \tag{21}$$

THE LAGRANGIAN FORMULATION

In the Lagrangian formation

$$\mathcal{L} = T(\dot{\mathbf{R}}, \dot{\phi}, \dot{r}, \dot{\varphi}_{\mathrm{A}}, \dot{\varphi}_{\mathrm{B}}) - \mathrm{V}(r, \varphi_{\mathrm{A}}, \varphi_{\mathrm{B}}), \tag{22}$$

therefore the coordinates $\mathbf{R}$ and ϕ are ignorable. The Lagrangian equations of motion

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{p}} = \frac{\partial\mathcal{L}}{\partial p}, \tag{23}$$

imply a constant of motion for each ignorable coordinate:

$$C_x = M\dot{R}_x, \tag{24a}$$
$$C_y = M\dot{R}_y, \tag{24b}$$
$$\ell = I_A(\dot{\phi} + \dot{\varphi}_A) + I_B(\dot{\phi} + \dot{\varphi}_B) + \mu r^2\dot{\phi} \tag{24c}$$

where ℓ is the total angular momentum. For simplicity take $C_1 = C_2 = 0$ and solve for $\dot{\phi}$ in terms of ℓ and the kinetic energy becomes

$$T = T_1 + T_2, \tag{25a}$$
$$T_1 = \frac{\frac{1}{2}\mu r^2(I_A\dot{\varphi}_A^2 + I_B\dot{\varphi}_B^2) + \frac{1}{2}I_AI_B(\dot{\varphi}_A - \dot{\varphi}_B)^2}{I_A + I_B + \mu r^2}, \tag{25b}$$
$$T_2 = \frac{\frac{1}{2}\ell^2 + \frac{1}{2}\mu\dot{r}^2(I_A + I_B + r^2)}{I_A + I_B + \mu r^2}. \tag{25c}$$

Therefore the simplified Lagrangian is

$$\mathcal{L} = T(r, \dot{r}, \dot{\varphi}_A, \dot{\varphi}_B) - V(r, \varphi_A, \varphi_B). \tag{26}$$

The final 3 Lagrange equations of motion must be numerically solved for a given set of initial conditions

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} = \frac{\partial\mathcal{L}}{\partial r}, \tag{27a}$$
$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\varphi_A}} = \frac{\partial\mathcal{L}}{\partial\varphi_A}, \tag{27b}$$
$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\varphi_B}} = \frac{\partial\mathcal{L}}{\partial\varphi_B}. \tag{27c}$$

FUTURE WORK

Future work includes simplifying the generalized radial force and torques,

$$\left\{\frac{d\mathcal{L}}{dr}, \frac{d\mathcal{L}}{d\varphi_A}, \frac{d\mathcal{L}}{d\varphi_B}\right\}, \tag{28}$$

in terms of the geometric quantities

$$\left\{\mathbf{r_{pp}}, \mathbf{r_{nn}}, \mathbf{r_{np}}, \mathbf{r_{pn}}, \hat{\mathbf{R}}_{\mathbf{A}}, \hat{\mathbf{R}}_{\mathbf{B}}\right\}. \tag{29}$$

Initial progress has been made.

A current weakness of the model is that the potential energy is undefined when the slashes are parallel, or $\Delta = 0$. The first idea to avoid this problem is to use numerical techniques to avoid the issue. Theoretically Δ will rarely be exactly zero so this method is plausible. Another idea would be to take the limit of the generalized torques and radial force as $\Delta \to 0$, and then when Δ is sufficiently small temporarily use the generalized radial force and torques limit formulas during the numerical integration. Further research would be needed to determine the best method.

I would like to thank Dr. Lindner for all his help and encouragement.

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