

Ångström's Method of Measuring Thermal Conductivity

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The thermal conductivity of a brass rod is measured using Ångström's method. A periodic square wave of heat is applied to a reservoir at one end of the rod, while the other end is left at room temperature. As the heat wave propagates down the rod, it is attenuated, losing amplitude and experiencing a phase shift. Two thermistors, located at separate points along the rod, measure the resulting temperature waves. Using a Fourier transform, the strongest contributions of the original square wave can be isolated and individually analyzed to deduce the thermal conductivity of the rod. Three values were obtained for the thermal conductivity, from the three strongest harmonics present in the observed temperature waves, $k=100\pm 3$, 120 ± 14 , and 180 ± 62 W/mK, in comparison with the accepted value of 120 W/mK.

INTRODUCTION

Measurement of the conduction of heat is a significant problem that affects many areas of material science and engineering. The applications of information on thermal conductivity range from the transfer of heat in solar panels to materials used in dentistry. Important applications to science and engineering include drawing heat from electrical components. One example of this is the mounting of semiconductors on diamonds to prevent damage from overheating, since diamonds have an extremely high conductivity at room temperature¹.

Many methods have been developed for measuring thermal conductivity accurately. Static methods involve measurements of the temperature gradient in conjunction with the heat flux. Dynamic methods are typically more effective at room temperature, heat losses having a smaller effect on the measurement¹. This experiment is based on a dynamic method of measuring thermal conductivity of a metal rod developed by Ångström in 1863. Heat is applied periodically to one end of a metal rod while the other end is left at the temperature of the surrounding medium. A heat wave propagates down the length of the rod, both losing amplitude and experiencing a phase shift. The fluctuations in temperature as a function of time are measured by two thermistors along the rod, and a comparison of the temperature waves leads to a determination of the thermal conductivity value for the metal.

THEORY

The derivation² of an expression for the thermal conductivity in terms of known or measurable quantities requires that the heat applied to the chamber be periodic rather than constant. Applying a periodic heat to one end of a rod produces a heat wave that propagates down the length of the rod, which can be measured by the difference in temperature. Expressing this temperature variation at any point of the rod becomes complicated due to the nature of the thermal processes within the rod; the change in heat is governed by several factors. There are four different processes considered that cause a change in the heat of the rod. One is the thermal conduction of heat, defined as the rate of heat energy lost through a surface ds , where k is the thermal conductivity (constant for a uniform material), and T is the temperature. Externally applying heat to a metal also increases its temperature, and the amount of heat required to change the temperature of a volume element dV of a metal of density ρ is defined by its specific heat s . Radiation accounts for heat lost through a surface element ds , which can be described by Newton's law of cooling, relating the temperature of the rod to its ambient temperature T_0 . R is a constant which depends on the radiative emission properties of the surface of the metal. The fourth way of changing the heat of the rod is a source within the rod. For the following experiment, there are no such sources, and so this process is not included in the expression.

Eqn. 1 shows the combination of the first three processes described above.

$$-\oint R(T - T_0)ds + \oint k T \cdot ds = \int s \frac{T}{t} dV \quad (1)$$

Using Green's theorem, the surface integral of the conductivity term may be changed to a volume integral. The integrals may be further reduced to position integrals in one dimension, integrating over the length of the rod from the near to the far thermistors. Since we are interested in relative and not absolute temperatures, we can express the temperature in terms of $T - T_0$, to give a wave equation for the temperature:

$$k \frac{\partial^2 T}{\partial x^2} - s \frac{\partial T}{\partial t} - \frac{PR}{A} = 0 \quad (2)$$

A solution to this equation, assuming the power applied is a pure sine wave of frequency $f = \omega/2\pi$, is

$$T - T_0 = B(x)e^{i\omega t} \quad (3)$$

in which T_0 is the mean temperature about which the oscillations occur, and ω is the frequency of the input wave. Substituting this back into Eqn. 2 and factoring out the $e^{i\omega t}$ gives a differential equation in x :

$$\frac{d^2 B(x)}{dx^2} = \left(\frac{PR}{Ak} + \frac{i s}{k} \right) B(x) \quad (4)$$

A complex solution is of the form

$$B(x) = B_0 e^{-\sqrt{\gamma} x} \quad (5)$$

where

$$\sqrt{\gamma} = \alpha + i\beta \quad (6)$$

The complex terms of the coefficient of Eqn. 4 and γ may be equated to develop an expression (Eqn. 7) for k independent of R , a value that is difficult to determine due to the nature of the experimental setup.

$$k = \frac{s}{2\alpha} \quad (7)$$

Determining expressions for α and β requires a comparison of the two temperature waves observed at x_1 and x_2 . The preceding discussion applies to a temperature wave that results from a sinusoidal input of power to the heater. For any other type of periodic input, such as a square wave, a Fourier transform may be used to isolate the individual contributions of the harmonics present in the temperature wave. For the n th harmonic, the temperature equation is (from Eqn. 3)

$$T(x, t) = A_n e^{i(n\omega t - n\alpha x)} \quad (8)$$

$$A_n = B_{0n} e^{-n\beta x}$$

The ratio of the amplitude observed at thermistor one (near) to that observed at thermistor two (far) can be rearranged to determine β :

$$\frac{A_{n1}}{A_{n2}} = e^{-n\beta(x_1 - x_2)} \quad \beta = \frac{\ln(A_{n1}/A_{n2})}{x_1 - x_2} \quad (9)$$

Similarly, we can find an expression for α by finding the difference in phase of the two temperature waves:

$$\alpha_1 - \alpha_2 = n(x_2 - x_1) \quad \alpha = \frac{(\phi_1 - \phi_2)}{n(x_2 - x_1)} \quad (10)$$

Substituting Eqns. 9 and 10 into Eqn. 7, gives the final expression for the thermal conductivity of the metal.

$$k = \frac{s n (x_2 - x_1)^2}{2(\alpha_2 - \alpha_1) \ln(A_{n1}/A_{n2})} \quad (11)$$

EXPERIMENT

A cylindrical brass rod, approximately 1m in length and 1cm in diameter, is suspended in the air; one end is supported by a shelf, the other end clamped to a ring stand. On one end is a thermofoil heater attached to a reservoir in thermal contact with the rod. The apparatus was originally designed to accommodate periodic heating by alternating currents of steam and cold water through the reservoir. Currently, the heating is accomplished by a Kepco Power Supply that applies a signal to the heater. The power supply is configured to send voltages of 0 to 50V. A Tektronix function generator provides the square wave signal to the power supply with a frequency of 10^{-3} s^{-1} . A very low frequency is used in order to allow the temperature wave to propagate through the rod with an amplitude that is easily measurable. The entire rod is also surrounded by both a foam insulation and bubble wrap in order to prevent loss of heat through radiation and conduction.

Two YSI 44004 Precision thermistors are housed in small holes drilled into the rod, and held in place by a thermally conductive epoxy. The distance between the thermistors, measured by a vernier caliper, is $15.0 \pm 0.1 \text{ cm}$. The thermistors are wired into a series circuit with a reference resistance of 15k Ω , and a 1.5V dry cell battery. The thermistors have a varying resistance that is related to their temperature, and the reference resistor provides a means of determining the current through the circuit.

Each resistor is connected to a channel of the Hewlett Packard 3421A Data Acquisition Unit, which measures the voltage across each one.

The collection of data is automated by a program written in LabVIEW 4.1 on a Macintosh. The program reads in the voltages from the HP3421A, and calculates the resistances of the thermistors. The LabVIEW program then calculates the temperature values and their corresponding time values, saving them in a text file.

The program is run to collect data until the rod has heated up to a constant mean temperature and the resulting wave has only its steady-state component for several periods. The actual frequency of the function generator must be chosen carefully in order to accommodate the analysis technique. The frequency must be adjusted so that there are 2^N points for an integral number of wavelengths of the temperature wave, in order to accommodate the Fourier transform.

The text file produced by the LabVIEW program contains the temperatures of the near and far thermistors as a function of time. The data used was collected from a single run. A small subsection of this data over four periods of the waves, which corresponds to about 90 minutes of data, was selected for analysis (see Figure 1). This subsection must be fairly consistent over time, well after the heating up of the rod, which took around four hours. The frequency ($\omega = (4.744 \pm 0.031) \times 10^{-3}$ Hz) of the wave was determined directly from the raw data. The change in time for several periods was divided into the number of wavelengths. The error was estimated from several such calculations.

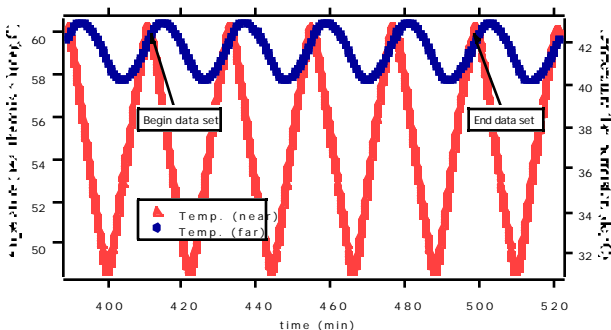


FIG. 1. The section of temperature data analyzed is displayed for both thermistors. Note the difference in amplitude and phase, but the identical frequency.

Figure 1 shows the temperature fluctuations measured by the two thermistors, and certain properties of how the wave propagates down the rod. When the

temperatures are graphed against time, it is clear that the temperature wave measured by the thermistor closer to the heater (near thermistor) has an amplitude much larger and a mean temperature higher than the thermistor located farther away (far thermistor). In addition to the difference in amplitude, the phase is slightly different for the two waves, with the wave at the near thermistor slightly trailing that of the far thermistor. Also, the frequency is the same for both waves, which corresponds to the frequency of the square wave input by the frequency generator to the heater.

The Fourier transform of the data was calculated by employing a function provided by Igor Pro. From the list of 1024 temperature points was calculated the Fourier transform of the wave, producing 513 real and imaginary points. Table I lists the magnitude and phase values of the Fourier transform for the first, third, and fifth harmonics. These were the strongest contributions of the terms of the Fourier series.

n	Near Temperature		Far Temperature	
	Mag.	Phase	Mag.	Phase
1	2620±70	0.0202±0.0021	700±10	1.22±0.01
3	210±10	0.120±0.015	26±5	2.09±0.13
5	57±5	0.0189±0.0089	6±4	2.02±0.32

Table I. The magnitude and phase values for the two waves, from the Fourier transform. n is the number of the harmonic.

The phase angles were observed to shift for the different harmonics. Calculation of the phase angle also required determining its quadrant, which was evident from the signs of the real and imaginary values. Since the arctan function returns values for only the first and fourth quadrants, radians were added to the second quadrant angles to shift their values to the correct quadrant.

Figure 2 shows the plot of the magnitudes of the Fourier transform versus point number, showing peaks at the odd harmonics. The number of the harmonic is the nth multiple of four in the point number values. This correspondence is a result of analysis over four wavelengths.

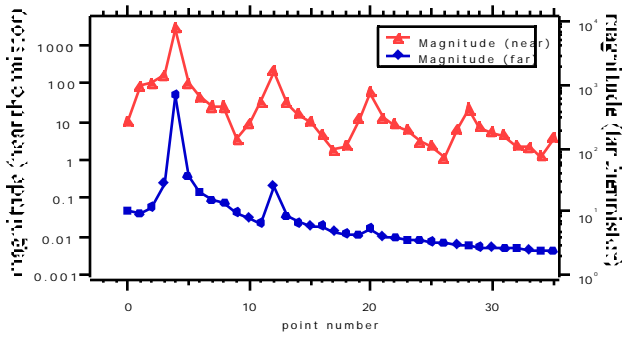


FIG 2. A graph of magnitude versus point number for the Fourier transform of both temperature waves. Peaks occur at the odd harmonics, stronger for the wave measured by the near thermistor than the far thermistor.

Figure 2 also exhibits certain properties of the temperature waves. The peaks for both thermistors occur at the odd harmonics. This is expected since the input to the heater was a square wave, which contains only odd harmonics. There are more peaks observed in the wave measured by the near thermistor than the far thermistor, indicating that there were stronger contributions from each harmonic of the input square wave measured at the near thermistor. At the far thermistor, the peak at the first harmonic is very significant, the peak at the third and fifth harmonics are much smaller in comparison, and beyond the fifth harmonic, any subsequent peaks are indistinguishable from the background noise. So the lowest frequencies were able to propagate farther along the rod than the higher frequencies. This is also evident in Figure 1, since the temperature wave at the far thermistor appears much closer to a sinusoidal wave than the wave at the near thermistor.

ANALYSIS AND INTERPRETATION

Values of k were calculated using accepted values of the density³ of brass as $\rho = 8470 \text{ kg/m}^3$, and the specific heat⁴ as $s = 368 \text{ J/kgK}$. There was some question as to what the type of brass was, but this uncertainty did not make a significant contribution to the calculation of error for the thermal conductivity values. Most of the error associated with the thermal conductivity values were a result of the uncertainties in the magnitude and phase.

Harmonic	$k(\text{W/mK})$	k
1	100	3
3	120	14
5	180	62
Accepted ³	120	--

Table II. The values for the error in the thermal conductivity of brass increase with the number of the harmonic as expected. The third and fifth harmonics have values for k within their errors, but the first harmonic does not.

The values for the error in the measured thermal conductivity increase for each successive harmonic. This reflects the uncertainty in the real and imaginary values for the Fourier transform as well. It is expected that the k value calculated from the first harmonic is the best known, since this is the strongest contribution in the waves measured at both thermistors. The third and fifth harmonics agree with the accepted value within their margins of error, the third harmonic showing quite close agreement. The first harmonic shows a significant difference from the accepted value, much larger than its error.

A number of possible sources of error in the thermal conductivity values are involved in the analysis. The Fourier transform executed by Igor required 2^N points per integral number of wavelengths for the section of data analyzed. It was difficult to collect data in exactly this manner, and so the actual section of data analyzed did not contain four entire wavelengths; there was a small section of the wavelength cut off in order to select the correct number of points. If the section of the temperature waves selected had mean temperature fluctuations, the Fourier transform values would also have been affected. Also, the uncertainties for the magnitudes and phase angles were chosen conservatively from the background noise, but may still not be representative of the actual uncertainty in these values.

CONCLUSION

The thermal conductivity of a brass rod at room temperature was measured using Ångström's method. Analysis of a temperature wave propagating down the rod using a Fourier transform revealed the individual contributions of the three strongest harmonics. For each harmonic, a value for the thermal conductivity was calculated. The thermal conductivity values from the third and fifth harmonics showed agreement with the accepted value within their error values, while the value from the first

harmonic had a very small error, but a much larger discrepancy from the accepted value.

¹J.E. Parrott and A.D. Stukes, Thermal Conductivity of Solids, (Pion Limited, London, 1975) p. **1**

² D.T. Jacobs, "Ångström's Method: Thermal Conductivity," (lecture notes, unpublished, 1999)

³David R. Lide, ed., Handbook of Chemistry and Physics, 77th ed. (CRC Press, New York, 1996) p. **12-185**

⁴ G.W.C. Kaye and T.H. Laby, Tables of Physical and Chemical Constants, 14th ed. (Longman, New York, 1973) p. **55**